

# SELF SIMILAR SOLUTIONS TO SUPER-CRITICAL GKDV

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ABSTRACT. We construct self similar finite energy solutions to the slightly super-critical generalized KdV equation. These self similar solutions bifurcate as a function of  $p$  from the soliton at the  $L^2$  critical exponent  $p = 4$ .

## 1. INTRODUCTION

Let  $p \geq 1$  and consider the generalized KdV equation

$$(1) \quad \begin{aligned} \partial_t u + \partial_{xxx} u \pm \partial_x(|u|^p u) &= 0 \\ u(0, x) &= u_0(x) \end{aligned}$$

or, for integer exponents  $p$ ,

$$(2) \quad \begin{aligned} \partial_t u + \partial_{xxx} u \pm \partial_x(u^{p+1}) &= 0 \\ u(0, x) &= u_0(x) \end{aligned}$$

Both the KdV equation ( $p = 1, (2)$ ) and the mKdV equation ( $p = 2, (2)$ ) are integrable and in these cases a remarkable amount of information can be obtained by the inverse scattering machinery. Both cases (1), (with  $+$ ) and (2) (either  $p$  odd or  $+$ ) admit soliton solutions  $u(x, t) = Q_p(x - t)$  where

$$(3) \quad Q_p(x) = \left( \frac{p+2}{2} \right)^{\frac{1}{p}} \operatorname{sech}^{\frac{2}{p}} \left( \frac{px}{2} \right).$$

The quantities

$$\int u dx, \quad \int u^2 dx \quad \text{and} \quad \int \frac{1}{2} u_x^2 - \frac{1}{p+2} |u|^{p+2} dx$$

are conserved. The equations are invariant under translations in space and time, and under the scaling

$$u_\lambda(x, t) = \lambda^{2/p} u(\lambda x, \lambda^3 t),$$

which shows that the homogeneous Sobolev spaces  $\dot{H}^s$  with index  $s = 1/2 - 2/p$  are scale invariant; the quintic gKdV equation ( $p = 4$ ) has  $L^2$  as critical space.

Kenig, Ponce and Vega [4] prove local existence to (2) in the scaling critical Sobolev space for all integers  $p \geq 4$  and global existence for  $p = 1, 2, 3$ . This has been extended to critical Besov spaces by Molinet and Ribaud in [15] and by Strunk [18] to (1) for real  $p > 4$ . This raises the question concerning global existence and blow up in the critical and the supercritical case  $p \geq 4$ . Numerical simulations by Dix and McKinney [3] suggest that there is self similar blow up in the supercritical case. In contrast to the situation for NLS there is neither a virial identity argument in the style of Glassey nor the explicit formula given by the pseudo-conformal transformation. Nonetheless, Martel and Merle, and Martel, Merle and Raphaël showed in a series of papers [10, 13, 11, 12, 8, 7, 9] that in the  $L^2$  critical case there are solutions which blow up along the soliton manifold, i.e. the spatial scale of the solution tends to zero in finite time. Probably one of the earliest and most prominent prediction of blow-up respectively wave collapse is due to Zacharov, Kuznetsov and Musher [21] for the (super critical) cubic focussing NLS

in three dimensions who write *Numerical simulations indicate that for  $d = 3$  there is self-similar and spherically symmetric blow-up, even from non-symmetric initial data*. The blow-up mechanism for the nonlinear Schrödinger equation is described in detail in the book by Sulem and Sulem [19]. Indeed, Zakharov [20] predicted blow-up of the form

$$\frac{1}{L(t)} \exp\left(\frac{1}{\tau(t)}\right) Q\left(\frac{|x|}{L(t)}; a\right)$$

for some selfsimilar profile  $Q$  and the scaling parameters

$$L(t) = (2a(t^* - t))^{1/2} \quad \text{and} \quad \tau(t) = \frac{1}{2a} \log\left(\frac{t^*}{t^* - t}\right),$$

where  $a > 0$  is a specific parameter and  $t^*$  is the time at which blow-up occurs. There seem to be solutions for each dimension  $2 < d \leq 3$  for one unique  $a(d)$  and heuristic arguments in [6] derive a relationship of the form

$$d - 2 \sim \frac{1}{a} \exp\left(-\frac{\pi}{a}\right).$$

It seems that the first fully rigorous construction of self-similar blow-up solutions is due to Kopell and Landman [5] for the cubic NLS in  $\mathbb{R}^{2+\varepsilon}$  (which has to be understood in the sense of existence of a solution to the nonlinear ODE into which the dimension enters merely as a parameter). It is crucial that these solutions are in  $\dot{H}^1 \cap L^{p+2}$  and hence their energy vanishes.

Self similar solutions for gKdV have been constructed by Bona and Weissler [1]. Their solutions are not in  $\dot{H}^1$  and their relation to the blow-up observed in simulations is not clear. Such solutions can be obtained by evolving small self-similar initial data, like for Navier-Stokes, or wave maps (Shatah et al).

In 2009, Merle, Raphael and Szeftel [14] established blow up from smooth initial data for NLS in the slightly super critical case in low dimensions, heuristically *bifurcating* from the soliton.

Here we construct selfsimilar solutions to the generalized KdV equation for  $p$  slightly larger than 4 (Theorem 2). Moreover in Theorem 3 we construct an almost invariant manifold containing the solitons and the selfsimilar solutions, which will play a central role in resolving the dynamic bifurcation at  $p = 4$ , together with fairly precise estimates in Theorem 1 for the constructed functions and their derivatives with respect to all parameters.

In Section 2 we formulate the bifurcation problem and state the main technical result, Theorem 1, and the main consequences, the existence result of Theorem 2 and Theorem 3. In Section 3 we deduce Theorem 2 and Theorem 3 from Theorem 1. This reduction is elementary but conceptually interesting.

The generalized Airy and Scorers functions are studied in Section 4 with standard arguments: Stationary phase, contour integrals, and explicit formulas of Fourier transforms of homogeneous functions. The next Section 5 derives explicit formulas for a unique Green's function for the linear part.

The next step (Section 6) consists in a study of estimates for integral operators with integral kernel related to the Green's function. After this preparation we set up the inverse function theorem in Section 6. Due to the weights differentiability with respect to  $a$  is not immediate. We approach it after establishing a fairly precise asymptotic expansion (Section 7) for the solution constructed by the inverse function theorem.

The final section shows plots of numerically computed self similar solutions for various values of  $a$  and  $p$  due to Strunk [17]. I want to thank Nils Strunk for allowing me to include this data and S. Steinerberger for many discussions.

## 2. THE BIFURCATION PROBLEM

We search for self-similar solutions  $\psi(t, x)$  of the form

$$(4) \quad \psi(t, x) = (3t)^{-\frac{2}{3p}} v \left( \frac{x}{(3t)^{1/3}} \right),$$

for which the self similar profile  $v$  has to satisfy

$$(5) \quad \frac{2}{p}v + yv_y - v_{yyy} - (|v|^p v)_y = 0.$$

A change of coordinates leads to a formulation in which the bifurcation from the soliton equation becomes visible: let  $a > 0$ ,

$$y = a^{1/3}(x + a^{-1}), \quad u(x) = a^{\frac{2}{3p}} v(a^{1/3}(x + a^{-1}))$$

then (5) is equivalent to

$$(6) \quad a \left( \frac{2}{p}u + xu_x \right) - u_{xxx} + u_x - (|u|^p u)_x = 0.$$

Reversing the derivation, if  $u$  satisfies (6) then

$$(7) \quad v(y) = a^{-\frac{2}{3p}} u(a^{-1/3}y - a^{-1})$$

is a solution for (5) and we thus get via (4) a self-similar solution for (1). We will construct self similar solutions in  $L^{p+2}$  with derivative in  $L^2$ . Since for any solution of gKdV the quantity

$$\int_{\mathbb{R}} \frac{1}{2}u_x^2 - \frac{1}{p+2}|u|^{p+2} dx$$

is formally conserved, plugging the ansatz into gKdV one sees that the existence of the integral already implies it being 0 for all times.

For  $a = 0$  the equation simplifies to the derivative of the soliton equation

$$(8) \quad \left( -u_{xx} + u - (|u|^p u) \right)_x = 0,$$

which motivates searching for a branch of solutions bifurcating from the soliton  $Q_p$  using  $a$  as bifurcation parameter (see also Sulem and Sulem [19]).

This is not yet the complete picture and complications arise from the linearization around the soliton

$$(9) \quad L\psi := -\psi_{xx} + \psi - (p+1)Q_p^p\psi$$

being elliptic but not invertible. Its spectrum, however, is explicitly known: there is a ground state  $Q_p^{\frac{p}{2}+1}$  and the second eigenvalue is 0 with an eigenspace spanned by  $Q'_p$ . We search  $p$  and  $u$  as functions of  $a$ . This requires an additional normalization which we choose to be

$$(10) \quad \langle u, Q'_p \rangle = 0.$$

Our considerations lead to the bifurcation formulation

$$(11) \quad a \left( \frac{2}{p}u + xu_x \right) - u_{xxx} + u_x - (|u|^p u)_x + \langle Q'_p, u \rangle Q_p'' = 0.$$

It will be useful to consider a generalization which will give an approximate invariant manifold which contains both, the solitons, and the selfsimilar blow up solutions. We consider

$$(12) \quad a((1+\gamma)u + xu_x) - u_{xxx} + u_x - (|u|^p u)_x + \langle Q'_p, u \rangle Q_p'' = 0.$$

**Theorem 1.** *Let  $q > 4$ . There exists  $\varepsilon > 0$  and a unique map*

$$u \in C^\infty \left( [0, \varepsilon) \times \left(-\frac{1}{2} - \varepsilon, -\frac{1}{2} + \varepsilon\right) \times (3, q) \times \mathbb{R} \right)$$

*with the following properties:*

$$(13) \quad u_{a,\gamma,p}(x) \text{ satisfies (12) for } 0 \leq a < \varepsilon, \left|\frac{1}{2} + \gamma\right| < \varepsilon, 2 \leq p \leq q$$

$$(14) \quad \sup_{a,\gamma,p,x} (1 + a|x|)^{1+\gamma} |u_{a,\gamma,p}(x)| < \infty,$$

$$(15) \quad \sup_{a,\gamma,p,x} (1 + x_+)^{1-k} |\partial_\gamma^n \partial_p^m \partial_a^k u_{a,\gamma,p}(x)| < \infty$$

*for  $k \geq 1$ ,*

$$(16) \quad u_{0,\gamma,p}(x) = Q_p(x)$$

$$(17) \quad u_{a,\gamma,p}(x) > 0, \quad \partial_x u_{a,\gamma,p} \in L^2(\mathbb{R})$$

*The solution  $u_{a,\gamma,p}$  is the unique solution to (12) satisfying (14) and (17) in a small neighborhood of the soliton.*

The main results are consequences.

**Theorem 2.** *There exists  $\varepsilon > 0$  and a unique function  $p \in C^\infty([0, \varepsilon))$  with*

$$p(0) = 4, \quad \frac{dp}{da}(0) = \frac{\|Q\|_{L^1}^2}{\|Q\|_{L^2}^2} = \frac{\Gamma(1/4)^4}{4\pi^2} \sim 4.3768 \dots$$

*such that  $x \rightarrow u_{a, \frac{2}{p(a)}-1, p(a)}(x)$  is a solution to (6) with*

$$(18) \quad \partial_x u_{a, \frac{2}{p(a)}-1, p(a)} \in L^2, \quad \sup(1 + a|x|)^{1+\gamma} |u_{a, p(a)}| < \infty$$

*and*

$$(19) \quad E(u_{a, \frac{2}{p(a)}-1, p(a)}) := \int \frac{1}{2} (\partial_x u_{a, \frac{2}{p(a)}-1, p(a)})^2 - \frac{1}{p+2} |u_{a, \frac{2}{p(a)}-1, p(a)}|^{p+2} dx = 0$$

These solutions are contained in a family of solutions which contains the solitons and the selfsimilar solution.

**Theorem 3.** *Let  $q > 4$ . There exists  $\varepsilon > 0$  and a unique function  $\gamma(a, p) \in C^\infty([0, \varepsilon) \times [3, q])$  with*

$$\begin{aligned} \gamma(0, 4) &= -\frac{1}{2}, \gamma(a, p(a)) = \frac{2}{p} - 1, \\ \frac{\partial \gamma}{\partial a}(0, 4) &= \frac{1}{8} \frac{\|Q\|_{L^1}^2}{\|Q\|_{L^2}^2} = \frac{1}{8} \frac{\Gamma(1/4)^4}{4\pi^2} \sim \frac{1}{8} 4.3768 \dots, \\ \frac{\partial \gamma}{\partial p}(0, 4) &= 0 \end{aligned}$$

*such that  $x \rightarrow u_{a, \gamma(a, p), p}(x)$  is a solution to*

$$(20) \quad a((1 + \gamma(a, p))u + xu_x) - (u_{xx} - u + |u|^p u)_x = 0$$

*with*

$$(21) \quad \partial_x u_{a, \gamma(a, p), p} \in L^2, \quad \sup(1 + a|x|)^{1+\gamma} |u_{a, \gamma(a, p), p}| \leq c.$$

*Moreover  $u_{0, \gamma(0, p), p} = Q_p$ .*

In the process of proving Theorem 1 we obtain fairly precise asymptotics for the constructed solutions. This asymptotics can be expressed concisely in terms of the special functions  $\text{Hi}_\gamma$  and  $\text{Gi}_\gamma$  constructed in Section 4.

## 3. THEOREM 1 IMPLIES THEOREM 2 AND THEOREM 3

3.1. **The soliton.** We recall that solitons  $Q$  satisfy, possibly after rescaling,

$$(22) \quad -Q_{xx} + Q - |Q|^p Q = 0.$$

There is a unique solution, up to the choice of sign and a translation parameter. We denote by  $Q$  (or  $Q_p$ ) the unique symmetric and nonnegative solution. We multiply by  $Q$  and  $xQ_x$ , respectively, and integrate to obtain the identities

$$(23) \quad \int Q_x^2 + Q^2 - Q^{p+2} dx = 0 = \int \frac{1}{2} Q_x^2 - \frac{1}{2} Q^2 + \frac{1}{p+2} Q^{p+2} dx.$$

This implies

$$(24) \quad \|Q\|_{L^{p+2}}^{p+2} = \frac{2(p+2)}{p+4} \|Q\|_{L^2}^2, \quad \|Q_x\|_{L^2}^2 = \frac{p}{p+4} \|Q\|_{L^2}^2$$

and hence

$$(25) \quad E(Q) = \int \frac{1}{2} Q_x^2 - \frac{1}{p+2} Q^{p+2} dx = \frac{p-4}{2(p+4)} \|Q\|_{L^2}^2$$

from which we see that the energy vanishes if  $p = 4$ . Let  $Q_c(x) = c^{-2/p} Q(x/c)$ , which is a rescaling of the soliton so that

$$(26) \quad u(x, t) = Q_c(x - c^2 t)$$

is a traveling wave solution to the gKdV equation with speed  $c^2$ . Then

$$\tilde{Q}_c := -c \frac{\partial}{\partial c} Q_c = \frac{2}{p} Q_c + \frac{x}{c} Q'_c$$

satisfies

$$\|Q_c\|_{L^2}^2 = c^{1-\frac{4}{p}} \|Q\|_{L^2}^2,$$

hence, using the notation  $\tilde{Q} = \tilde{Q}_1$ ,

$$(27) \quad \langle \tilde{Q}, Q \rangle = -\frac{1}{2} \frac{d}{dc} \|Q_c\|_{L^2}^2 \Big|_{c=1} = \left( \frac{2}{p} - \frac{1}{2} \right) \|Q\|_{L^2}^2$$

which changes sign as  $p$  passes through 4. We differentiate (26) with respect to  $c$ , evaluate at  $c = 1$  and obtain a solution to the linearized equation, hence

$$\frac{d}{dx} (-2Q - L\tilde{Q}) = 0$$

and

$$(28) \quad L\tilde{Q} = -2Q.$$

An integration by parts gives

$$(29) \quad \int \tilde{Q} dx = \int \frac{2}{p} Q + x Q_x dx = \left( \frac{2}{p} - 1 \right) \int Q dx.$$

3.2. **The derivatives with respect to  $a$ .** Let  $\dot{v}$  be the derivative of  $u$  with respect to  $a$  evaluated at  $a = 0$ . It decays at  $-\infty$  and hence it satisfies

$$(30) \quad L\dot{v} + \langle \dot{v}, Q_x \rangle Q_x = - \int_{-\infty}^x (1 + \gamma) Q + x Q_x dy$$

We multiply by  $Q_x$  (supressing  $p$  in the notation) and, since  $LQ_x = 0$ , and

$$(31) \quad \langle \dot{v}, Q_x \rangle = \|Q_x\|_{L^2}^{-2} \langle Q, \tilde{Q} \rangle = \left( \gamma + \frac{1}{2} \right) \frac{\|Q\|_{L^2}^2}{\|Q_x\|_{L^2}^2}.$$

Observe that  $\langle \dot{v}, Q_x \rangle = 0$  if  $\gamma = -\frac{1}{2}$ . The norms on the right hand side can be evaluated and this gives the derivative of the inner product with respect to  $a$  at  $a = 0$  as a function of  $p$  and  $\gamma$ .

We set  $\gamma = -\frac{1}{2}$ , multiply (30) by  $\dot{v}$  and integrate

$$\langle \frac{1}{2}Q + xQ_x, \dot{v} \rangle + \int \dot{v}_x \dot{v} dx + (p+1)\langle Q^p \dot{v}, \dot{v}_x \rangle = 0.$$

We rewrite the middle integral as a limit

$$\int \dot{v}_x \dot{v} dx = \lim_{R \rightarrow \infty} \int_{-\infty}^R \dot{v}_x \dot{v} dx = \lim_{R \rightarrow \infty} \frac{1}{2}(\dot{v}(R))^2$$

This limit can be calculated: the inverse of  $-\partial_{xx} + 1$  is given by the convolution by  $\frac{1}{2}e^{-|x|}$ . It maps the constant function 1 to itself, hence

$$\lim_{R \rightarrow \infty} \frac{1}{2}(\dot{v}(R))^2 = \frac{1}{2}(\int \tilde{Q} dx)^2 = \frac{1}{8} \left( \int Q dx \right)^2$$

and

$$(32) \quad \langle \frac{1}{2}Q + xQ_x, \dot{v} \rangle + (p+1)\langle Q^p \dot{v}, \dot{v}_x \rangle = -\frac{1}{8} \left( \int Q dx \right)^2.$$

Let  $\ddot{v}$  be the second derivative with respect to  $a$  evaluated at  $a = 0$ . It satisfies

$$2(\frac{1}{2}\dot{v} + x\partial_x \dot{v}) + \partial_x (L\ddot{v} - p(p+1)Q^{p-1}\dot{v}^2 + \langle \ddot{v}, Q_x \rangle Q_x) = 0$$

We fix  $p = 4$ , multiply by  $Q$  and integrate. Then, since

$$\langle \dot{v}, \frac{1}{2}Q + xQ_x \rangle + \langle \frac{1}{2}\dot{v} + x\dot{v}_x, Q \rangle = 0,$$

and using (32),

$$\begin{aligned} \|Q_x\|_{L^2}^2 \langle \ddot{v}, Q_x \rangle &= 2\langle \frac{1}{2}\dot{v} + x\partial_x \dot{v}, Q \rangle + 20 \int Q^3 Q_x \dot{v}^2 dx \\ &= -2\langle \dot{v}, \frac{1}{2}Q + xQ_x \rangle - 10 \int Q^4 \dot{v} \dot{v}_x dx \\ &= \frac{1}{4} \|Q\|_{L^1}^2 \end{aligned}$$

and hence the second derivative of the inner product with respect to  $a$  at  $a = 0$ ,  $\gamma = -\frac{1}{2}$  and  $p = 4$  is given by

$$(33) \quad \langle \ddot{v}, Q_x \rangle = \frac{1}{4} \frac{\|Q\|_{L^1}^2}{\|Q_x\|_{L^2}^2}.$$

We define the smooth function

$$(a, \gamma, p) \rightarrow \eta(a, \gamma, p) := \langle u_{a, \gamma, p}, \partial_x Q_p \rangle$$

on  $[0, \varepsilon) \times (-\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon) \times [2, q]$ . The orthogonality  $\langle Q, Q_x \rangle = 0$  implies

$$\eta(0, \gamma, p) = 0,$$

the derivative with respect to  $a$  is given by (31)

$$\frac{\partial \eta}{\partial a}(0, \gamma, p) = \left( \gamma + \frac{1}{2} \right) \frac{\|Q\|_{L^2}^2}{\|Q_x\|_{L^2}^2},$$

hence

$$(34) \quad \frac{\partial \eta}{\partial a}(0, -\frac{1}{2}, p) = 0$$

and

$$\frac{\partial^2 \eta}{\partial a \partial \gamma}(0, \gamma, p) = \frac{\|Q\|_{L^2}^2}{\|Q_x\|_{L^2}^2}.$$

We read the second derivative with respect to  $a$  from (33)

$$(35) \quad \frac{\partial^2 \eta}{\partial a^2}(0, -\frac{1}{2}, 4) = \frac{1}{4} \frac{\|Q\|_{L^1}^2}{\|Q_x\|_{L^2}^2}.$$

Let

$$(36) \quad g(a, \gamma, p) = \eta(a, \gamma, p)/a$$

which is a smooth function with (again we suppress  $p$  in the notation of  $Q$ )

$$\begin{aligned} g(0, \gamma, p) &= 0, \\ \frac{\partial g}{\partial a}(0, -\frac{1}{2}, 4) &= \frac{1}{8} \frac{\|Q\|_{L^1}^2}{\|Q_x\|_{L^2}^2} \\ \frac{\partial g}{\partial \gamma}(0, -\frac{1}{2}, p) &= \frac{\|Q\|_{L^2}^2}{\|Q_x\|_{L^2}^2} \\ \frac{\partial g}{\partial p}(0, -\frac{1}{2}, p) &= 0 \end{aligned}$$

and by the implicit function theorem the equation

$$g(a, \frac{2}{p} - 1, p) = 0$$

can be solved for  $p = p(a)$  for  $a \in [0, \varepsilon)$ , possibly after decreasing  $\varepsilon$  if necessary. Clearly  $p(0) = 4$  and

$$\frac{dp}{da}(0) = - \frac{\frac{\partial g}{\partial a}}{\frac{\partial g}{\partial p}} \bigg|_{a=0, p=4} = - \frac{\frac{1}{2} \frac{\partial^2 \eta}{\partial a^2}}{\frac{\partial^2 \eta}{\partial a \partial p}} \bigg|_{a=0, p=4} = \frac{\|Q\|_{L^1}^2}{\|Q\|_{L^2}^2}$$

We recall

$$Q(x) = 3^{1/4} \operatorname{sech}^{1/2}(2x)$$

and thus

$$\begin{aligned} \int Q dx &= \frac{3072^{1/4}}{\sqrt{\pi}} \Gamma\left(\frac{5}{4}\right)^2, \\ \int Q^2 dx &= \frac{\sqrt{3}}{2} \pi \end{aligned}$$

and hence

$$\frac{dp}{da}(0) = \frac{64}{\pi^2} \Gamma\left(\frac{5}{4}\right)^4 = \frac{1}{4\pi^2} \Gamma(1/4)^4 \sim 4.3768 \dots$$

The changes for Theorem 3 are quite obvious: We solve

$$g(a, \gamma, p) = 0$$

for  $\gamma(a, p)$  near  $(0, -\frac{1}{2}, 4)$  and obtain

$$\frac{\partial}{\partial a} \gamma(0, 4) = \frac{1}{32\pi^2} \Gamma(1/4)^4 \sim 0.54711$$

and

$$\frac{\partial}{\partial p} \gamma(0, 4) = 0.$$

**3.3. Vanishing energy.** The function  $u_{a, \frac{2}{p(a)}-1, p(a)}$  satisfies (11) and (10), hence (6). Moreover

$$|u_{a, \frac{2}{p(a)}-1, p(a)}| \leq c(1 + |x|)^{-2/p} \in L^{p+2}$$

for  $p \geq 1$ . By Theorem 1 the derivative with respect to  $x$  is in  $L^2$ . We observed above that then the energy has to vanish.

This completes the proof that Theorem 1 implies Theorem 2 and Theorem 3.

## 4. THE AIRY FUNCTION AND SCORER'S FUNCTIONS

**4.1. Definition and first properties.** In this section we study a class of special functions closely related to the Airy function. The Airy function and Scorer's functions are discussed in [16], and the notation is motivated by Dix [2] but with deliberate essential changes. We define for  $\gamma \in \mathbb{C}$  with real part larger than  $-1$

$$\begin{aligned} \text{Ai}_\gamma(x) &= \frac{1}{2\pi} \text{real} \int_{-\infty}^{\infty} (\sigma/i)^\gamma e^{i(\sigma^3/3 + x\sigma)} d\sigma \\ &= \frac{1}{\pi} \int_0^{\infty} \sigma^\gamma \cos\left(\frac{1}{3}\sigma^3 + x\sigma - \frac{\gamma\pi}{2}\right) d\sigma \\ &= \frac{1}{\pi} \int_0^{\infty} \sigma^\gamma \left( \cos\left(\frac{\gamma\pi}{2}\right) \cos\left(\frac{1}{3}\sigma^3 + x\sigma\right) + \sin\left(\frac{\gamma\pi}{2}\right) \sin\left(\frac{1}{3}\sigma^3 + x\sigma\right) \right) d\sigma. \end{aligned}$$

Clearly  $\text{Ai}_\gamma$  depends holomorphically on  $\gamma$ .

The first line of the equation defines  $\text{Ai}_\gamma$  through the Fourier transform. The second line is the corresponding real formulation (if  $\gamma$  is real) and the last line connects the definition to the slightly different ones in [2]. We easily see that

$$(37) \quad \text{Ai}'_\gamma = -A_{\gamma+1}$$

and

$$(38) \quad (1 + \gamma) \text{Ai}_\gamma + x \text{Ai}'_\gamma - \text{Ai}'''_\gamma = 0.$$

This identity can be rewritten as

$$(39) \quad (1 + \gamma) \text{Ai}_\gamma - x \text{Ai}_{\gamma+1} + \text{Ai}_{\gamma+3} = 0,$$

moreover,

$$\text{Ai}_0 = \text{Ai}.$$

It is not hard to evaluate the function  $\text{Ai}_\gamma$  at  $x = 0$

$$\begin{aligned} \text{Ai}_\gamma(0) &= \frac{1}{\pi} \text{real} \int_0^{\infty} (\sigma/i)^\gamma e^{i\sigma^3/3} d\sigma \\ (40) \quad &= \frac{1}{\pi} 3^{\frac{\gamma-2}{3}} e^{-\frac{(\gamma-2)\pi}{3}} \text{Im} \int_0^{\infty} \mu^{\frac{\gamma-2}{3}} e^{-\mu} d\mu \\ &= -\frac{1}{\pi} \sin\left(\frac{1}{3}\pi(\gamma-2)\right) 3^{\frac{\gamma-2}{3}} \Gamma((\gamma+1)/3). \end{aligned}$$

We work out the asymptotic behavior using the standard approach via contour integration and stationary phase. If  $x < 0$  we apply stationary phase and shift the contour around zero to the upper half plane so that the leading contribution comes from the stationary point  $\xi = (-x)^{1/3}$ . We obtain the leading term

$$(41) \quad \text{Ai}_\gamma \sim \frac{1}{\sqrt{\pi}} |x|^{-\frac{1}{4} + \frac{\gamma}{2}} \cos\left(\frac{2}{3}|x|^{3/2} - \frac{\pi}{4} - \frac{\gamma\pi}{2}\right)$$

as  $x \rightarrow -\infty$ . More precisely

$$(42) \quad \text{Ai}_\gamma(x) = \text{real} \left\{ \left( \frac{1}{\sqrt{\pi}} |x|^{-\frac{1}{4} + \frac{\gamma}{2}} + O(|x|^{-\frac{7}{4} + \frac{\gamma}{2}}) \right) e^{i(\frac{2}{3}|x|^{3/2} - \frac{\pi}{4} - \frac{\gamma\pi}{2})} \right\}$$

as  $x \rightarrow -\infty$  and we can replace  $O(|x|^{-\frac{7}{4} + \frac{\gamma}{2}})$  by an asymptotic series

$$(43) \quad |x|^{-\frac{7}{4} + \frac{\gamma}{2}} \sum_{j=0}^{\infty} c_j |x|^{-3j/2}.$$



We turn to  $x > 0$ , shift the contour of integration to  $\mathbb{R} + i\sqrt{x}$  and obtain again by stationary phase

$$(44) \quad \text{Ai}_\gamma = \left( \frac{1}{2\sqrt{\pi}} |x|^{-\frac{1}{4} + \frac{\gamma}{2}} + O(|x|^{-\frac{7}{4} + \frac{\gamma}{2}}) \right) e^{-\frac{2}{3}x^{3/2}}$$

as  $x \rightarrow \infty$ . Again the  $O(|x|^{-\frac{7}{4} + \frac{\gamma}{2}})$  can be replaced by an asymptotic series (43). These series can be differentiated term by term with respect to  $\gamma$ , with the expected estimates for the difference of  $\text{Ai}_\gamma$  to the partial sum.

Similarly, we set for the same set of  $\gamma$

$$\begin{aligned} \text{Gi}_\gamma(x) &= \frac{1}{\pi} \text{Im} \int_0^\infty (\sigma/i)^\gamma e^{i(\frac{1}{3}\sigma^3 + x\sigma)} d\sigma \\ &= \frac{1}{\pi} \int_0^\infty \sigma^\gamma \sin\left(\frac{1}{3}\sigma^3 + x\sigma - \frac{\gamma\pi}{2}\right) d\sigma \\ &\quad - \frac{1}{\pi} \int_0^\infty \sigma^\gamma \left( -\sin\left(\frac{\gamma\pi}{2}\right) \cos\left(\frac{1}{3}\sigma^3 + x\sigma\right) + \cos\left(\frac{\gamma\pi}{2}\right) \sin\left(\frac{1}{3}\sigma^3 + x\sigma\right) \right) d\sigma. \end{aligned}$$

Again it is easily seen that

$$\text{Gi}'_\gamma = -\text{Gi}_{\gamma+1}$$

and

$$(1 + \gamma) \text{Gi}_\gamma + x \text{Gi}'_\gamma - \text{Gi}'''_\gamma = 0$$

which we can again rewrite as

$$(1 + \gamma) \text{Gi}_\gamma - x \text{Gi}_{\gamma+1} + \text{Gi}_{\gamma+3} = 0.$$

Evaluation at zero gives

$$(45) \quad \text{Gi}_\gamma(0) = \frac{1}{\pi} \text{Im} \int_0^\infty (\sigma/i)^\gamma e^{i\frac{1}{3}\sigma^3} d\sigma = -\frac{1}{\pi} \cos\left(\frac{\pi}{3}(\gamma - 2)\right) 3^{\frac{\gamma-2}{3}} \Gamma((\gamma + 1)/3)$$

There are two contributions for large  $x$ , one from the integral near zero and a second one from the oscillatory part. We choose a smooth cutoff function supported in  $|\sigma| \leq 2$ , identically 1 in  $|\sigma| \leq 1$  and we write

$$\begin{aligned} \text{Gi}_\gamma(x) &= \text{Gi}_\gamma^s + \text{Gi}_\gamma^0 \\ &= \frac{1}{\pi} \text{Im} \int_0^\infty \eta(\sigma) (\sigma/i)^\gamma e^{i(\frac{1}{3}\sigma^3 + x\sigma)} d\sigma \\ &\quad + \frac{1}{\pi} \text{Im} \int_0^\infty (1 - \eta(\sigma)) (\sigma/i)^\gamma e^{i(\frac{1}{3}\sigma^3 + x\sigma)} d\sigma. \end{aligned}$$

Then

$$\begin{aligned} \text{Gi}_\gamma^s(x) &= \sum_{j=0}^\infty \frac{1}{j!} (-1/3)^j \frac{1}{\pi} \text{Im} \int_0^\infty (\sigma/i)^{\gamma+3j} e^{ix\sigma} \eta(\sigma) d\sigma \\ (46) \quad &= \sum_{j=0}^\infty \frac{(-1/3)^j}{\pi j!} \int_0^\infty (\sigma/i)^{\gamma+3j} e^{ix\sigma} d\sigma + O(|x|^{-\infty}). \end{aligned}$$

in the sense of oscillatory integrals. Now suppose that  $x > 0$ . Then we move the contour of integration to  $i\mathbb{R}_+$ :

$$(47) \quad \int_0^\infty (\sigma/i)^\mu e^{ix\sigma} d\sigma = \int_{i\mathbb{R}_+} (\sigma/i)^\mu e^{ix\sigma} d\sigma = i \int_0^\infty t^\mu e^{-xt} dt = ix^{-1-\mu} \Gamma(1 + \mu)$$

If  $x < 0$  we move the contour to  $-i\mathbb{R}_+$  and obtain for  $\text{Gi}_\gamma^s(x)$

$$(48) \quad \sum_{j=0}^\infty \frac{(-1/3)^j \Gamma(1 + \gamma + 3j)}{j! \pi} |x|^{-1-\gamma-3j} \begin{cases} -\cos(\pi(\gamma + 3j)) & x < 0 \\ 1 & x > 0 \end{cases} + O(|x|^{-\infty}).$$

The oscillatory part (for  $x < 0$ ) is dealt with as above and we obtain the leading term

$$(49) \quad \text{Gi}_\gamma^o \sim -\frac{1}{\sqrt{\pi}}|x|^{-\frac{1}{4}+\gamma/2} \sin\left(\frac{2}{3}|x|^{3/2} - \frac{\pi}{4} - \frac{\gamma\pi}{2}\right)$$

as  $x \rightarrow -\infty$ , again with the same type of asymptotic series, and it is  $O(|x|^{-\infty})$  as  $x \rightarrow \infty$ . Again it can be differentiated term by term with respect to  $x$  and  $\gamma$ .

Finally, we set for  $\gamma > -1$

$$\text{Hi}_\gamma(x) = \frac{1}{\pi} \int_0^\infty \sigma^\gamma e^{-\frac{1}{3}\sigma^3 + \sigma x} d\sigma.$$

The derivative is again simple

$$\text{Hi}_\gamma' = \text{Hi}_{\gamma+1}$$

and furthermore

$$(1 + \gamma) \text{Hi}_\gamma + x \text{Hi}_\gamma' - \text{Hi}_\gamma''' = 0$$

which we rewrite as

$$(1 + \gamma) \text{Hi}_\gamma + x \text{Hi}_{\gamma+1} - \text{Hi}_{\gamma+3} = 0$$

The evaluation at  $x = 0$  is given by

$$(50) \quad \text{Hi}_\gamma(0) = \frac{1}{\pi} \int_0^\infty \sigma^\gamma e^{-\sigma^3/3} d\sigma = \frac{1}{\pi} 3^{\frac{\gamma-2}{3}} \int \rho^{(\gamma-2)/3} e^{-\rho} d\rho = \frac{1}{\pi} 3^{\frac{\gamma-2}{3}} \Gamma((\gamma+1)/3).$$

It is not hard to see that

$$\text{Hi}_\gamma(x) = \sum_{j=0}^\infty \frac{\Gamma(1 + \gamma + 3j)}{3^j j! \pi} |x|^{-1-\gamma-3j} + O(|x|^{-\infty})$$

as  $x \rightarrow -\infty$  and

$$(51) \quad \text{Hi}_\gamma(x) = \left\{ \frac{1}{\sqrt{\pi}} x^{-\frac{1}{4}+\frac{\gamma}{2}} + O(x^{-\frac{7}{4}+\frac{\gamma}{2}}) \right\} e^{\frac{2}{3}x^{3/2}}$$

as  $x \rightarrow \infty$ , where again the  $O(|x|^{-\frac{7}{4}+\frac{\gamma}{2}})$  terms can be sharpened to an asymptotic series. The functions  $H_\gamma$  and all their  $x$  derivatives are nonnegative. Derivatives with respect to  $x$  and  $\gamma$  can be handled as above.

**4.2. Wronskian determinant.** The three functions  $\text{Ai}_\gamma$ ,  $\text{Gi}_\gamma$  and  $\text{Hi}_\gamma$  satisfy the same differential equation. Here we will collect properties of the Wronskian matrix defined by those functions.

The Wronskian determinant  $W$  is independent of  $x$  since there is no second derivative in the ODE and we evaluate it at  $x = 0$

$$\begin{aligned} W &= \det \begin{pmatrix} \text{Ai}_\gamma(0) & \text{Gi}_\gamma(0) & \text{Hi}_\gamma(0) \\ -\text{Ai}_{\gamma+1}(0) & -\text{Gi}_{\gamma+1}(0) & \text{Hi}_{\gamma+1}(0) \\ \text{Ai}_{\gamma+2}(0) & \text{Gi}_{\gamma+2}(0) & \text{Hi}_{\gamma+2}(0) \end{pmatrix} \\ &= \pi^{-3} 3^{\gamma-1} \Gamma\left(\frac{\gamma+1}{3}\right) \Gamma\left(\frac{\gamma+2}{3}\right) \Gamma\left(\frac{\gamma+3}{3}\right) \det \begin{pmatrix} \sin\left(\frac{(\gamma-2)\pi}{3}\right) & \cos\left(\frac{(\gamma-2)\pi}{3}\right) & 1 \\ -\sin\left(\frac{(\gamma-1)\pi}{3}\right) & -\cos\left(\frac{(\gamma-1)\pi}{3}\right) & 1 \\ \sin\left(\frac{\gamma\pi}{3}\right) & \cos\left(\frac{\gamma\pi}{3}\right) & 1 \end{pmatrix} \end{aligned}$$

The Gaussian multiplication formula simplifies the product of the  $\Gamma$  functions

$$\Gamma\left(\frac{\gamma+1}{3}\right) \Gamma\left(\frac{\gamma+2}{3}\right) \Gamma\left(\frac{\gamma+3}{3}\right) = 2\pi 3^{-1/2-\gamma} \Gamma(1 + \gamma).$$

The remaining determinant can be expanded and simplified via addition theorems and evaluates to  $3\sqrt{3}/2$ .

Altogether, we arrive at

$$(52) \quad W = \frac{\Gamma(\gamma+1)}{\pi^2}.$$

In particular, the functions  $\text{Ai}_\gamma$ ,  $\text{Gi}_\gamma$  and  $\text{Hi}_\gamma$  are a fundamental system for the differential equation

$$(53) \quad (1 + \gamma)u + xu_x - u_{xxx} = 0.$$

**4.3. Subdeterminants.** Let

$$f(x) = \text{Ai}_\gamma \text{Gi}'_\gamma - \text{Ai}'_\gamma \text{Gi}_\gamma = -\text{Ai}_\gamma \text{Gi}_{\gamma+1} + \text{Ai}_{\gamma+1} \text{Gi}_\gamma =: [\text{Ai}_\gamma, \text{Gi}_\gamma]$$

and calculate

$$\begin{aligned} xf' - f''' &= x \left( \text{Ai}_\gamma \text{Gi}_{\gamma+2} - \text{Ai}_{\gamma+2} \text{Gi}_\gamma \right) \\ &\quad - \text{Ai}_\gamma \text{Gi}_{\gamma+4} - 2 \text{Ai}_{\gamma+1} \text{Gi}_{\gamma+3} + 2 \text{Ai}_{\gamma+3} \text{Gi}_{\gamma+1} + \text{Ai}_{\gamma+4} \text{Gi}_\gamma \\ &= \text{Ai}_\gamma (x \text{Gi}_{\gamma+2} - \text{Gi}_{\gamma+4}) - \text{Gi}_\gamma (x \text{Ai}_{\gamma+2} - \text{Ai}_{\gamma+4}) \\ &\quad + 2 \text{Ai}_{\gamma+1} (x \text{Gi}_{\gamma+1} - \text{Ai}_{\gamma+3}) - 2 \text{Gi}_{\gamma+1} (x \text{Ai}_{\gamma+1} - \text{Ai}_{\gamma+3}) \\ &= (2 + \gamma) (\text{Ai}_\gamma \text{Gi}_{\gamma+1} - \text{Ai}_{\gamma+1} \text{Gi}_\gamma) + 2(1 + \gamma) (\text{Ai}_{\gamma+1} \text{Gi}_\gamma - \text{Ai}_\gamma \text{Gi}_{\gamma+1}) \\ &= -\gamma (\text{Ai}_\gamma \text{Gi}_{\gamma+1} - \text{Ai}_{\gamma+1} \text{Gi}_\gamma) \\ &= -(1 + \tilde{\gamma})f \end{aligned}$$

with

$$\tilde{\gamma} = -1 - \gamma.$$

Hence

$$f = c_1 \text{Ai}_{\tilde{\gamma}} + c_2 \text{Gi}_{\tilde{\gamma}} + c_3 \text{Hi}_{\tilde{\gamma}}.$$

The function  $f$  heritates the faster than polynomial decay for  $x \gg 1$  from  $\text{Ai}_\gamma$ . Thus  $c_2 = c_3 = 0$ . The leading term to the right is

$$\frac{1}{2\sqrt{\pi}} x^{\frac{1}{4} + \frac{\gamma}{2}} \frac{\Gamma(1 + \gamma)}{\pi} x^{-1 - \gamma} e^{-\frac{2}{3}x^{\frac{3}{2}}}$$

We compare this with the asymptotic of  $\text{Ai}_{-1-\gamma}$  which gives

$$(54) \quad [\text{Ai}_\gamma, \text{Gi}_\gamma] = \frac{\Gamma(1 + \gamma)}{\pi} \text{Ai}_{-1-\gamma}$$

Similarly

$$[\text{Ai}_\gamma, \text{Hi}_\gamma] = c_1 \text{Ai}_{\tilde{\gamma}} + c_2 \text{Gi}_{\tilde{\gamma}} + c_3 \text{Hi}_{\tilde{\gamma}}$$

The leading term for  $x \gg 1$  is

$$\frac{1}{\pi} x^\gamma$$

and hence

$$c_3 = 0, \quad c_2 = \frac{1}{\Gamma(-\gamma)}.$$

We recall that

$$\Gamma(1 - s)\Gamma(s) = \frac{\pi}{\sin(s\pi)}.$$

to rewrite

$$c_2 = -\frac{\Gamma(1 + \gamma)}{\pi} \sin(\gamma\pi)$$

The leading term for  $x \ll -1$  is

$$\frac{\Gamma(1 + \gamma)}{2\pi^{3/2}} |x|^{-\frac{3}{4} - \frac{\gamma}{2}} \cos\left(\frac{2}{3}|x|^{3/2} - \frac{\pi}{4} - \frac{(\gamma + 1)\pi}{2}\right)$$

where

$$\begin{aligned} \cos\left(\frac{2}{3}|x|^{3/2} - \frac{\pi}{4} - \frac{(\gamma + 1)\pi}{2}\right) &= \cos\left(\frac{2}{3}|x|^{3/2} - \frac{\pi}{4} - \frac{\tilde{\gamma}\pi}{2}\right) \cos((\gamma + 1)\pi) \\ &\quad + \sin\left(\frac{2}{3}|x|^{3/2} - \frac{\pi}{4} - \frac{\tilde{\gamma}\pi}{2}\right) \sin((\gamma + 1)\pi) \end{aligned}$$

hence

$$c_1 = -\frac{\Gamma(1+\gamma)}{\pi} \cos(\gamma\pi)$$

$$(55) \quad [\text{Ai}_\gamma, \text{Hi}_\gamma] = -\frac{\Gamma(1+\gamma)}{\pi} \left( \cos(\gamma\pi) \text{Ai}_{\tilde{\gamma}} + \sin(\gamma\pi) \text{Gi}_{\tilde{\gamma}} \right).$$

Finally

$$[\text{Gi}_\gamma, \text{Hi}_\gamma] = c_1 \text{Ai}_{\tilde{\gamma}} + c_2 \text{Gi}_{\tilde{\gamma}} + c_3 \text{Hi}_{\tilde{\gamma}}.$$

The leading term for  $x \gg 1$  is

$$\frac{\Gamma(1+\gamma)}{\pi^{3/2}} |x|^{-\frac{3}{4}-\frac{\gamma}{2}} e^{\frac{2}{3}x^{\frac{3}{2}}}$$

hence

$$(56) \quad c_3 = \frac{\Gamma(1+\gamma)}{\pi}.$$

The leading oscillatory term for  $x \ll -1$  is

$$\frac{\Gamma(1+\gamma)}{2\pi^{3/2}} |x|^{-\frac{3}{4}-\frac{\gamma}{2}} \sin\left(\frac{2}{3}|x|^{3/2} - \frac{\pi}{4} - \frac{(\gamma+1)\pi}{2}\right)$$

where

$$\begin{aligned} \sin\left(\frac{2}{3}|x|^{3/2} - \frac{\pi}{4} - \frac{(\gamma+1)\pi}{2}\right) &= \sin\left(\frac{2}{3}|x|^{3/2} - \frac{\pi}{4} - \frac{\tilde{\gamma}\pi}{2}\right) \cos((\gamma+1)\pi) \\ &\quad - \cos\left(\frac{2}{3}|x|^{3/2} - \frac{\pi}{4} - \frac{\tilde{\gamma}\pi}{2}\right) \sin((\gamma+1)\pi) \end{aligned}$$

hence

$$c_1 = \frac{\Gamma(1+\gamma)}{\pi} \sin(\gamma\pi), \quad c_2 = -\frac{\Gamma(1+\gamma)}{\pi} \cos(\gamma\pi)$$

and

$$(57) \quad [\text{Gi}_\gamma, \text{Hi}_\gamma] = \frac{\Gamma(1+\gamma)}{\pi} \left( \sin(\gamma\pi) \text{Ai}_{\tilde{\gamma}} - \cos(\gamma\pi) \text{Gi}_{\tilde{\gamma}} + \text{Hi}_{\tilde{\gamma}} \right)$$

We collect all the formulas in a proposition.

**Proposition 4.** *The following identities hold*

$$\begin{aligned} [\text{Ai}_\gamma, \text{Gi}_\gamma] &= \frac{\Gamma(1+\gamma)}{\pi} \text{Ai}_{-1-\gamma} \\ (58) \quad [\text{Ai}_\gamma, \text{Hi}_\gamma] &= \frac{\Gamma(1+\gamma)}{\pi} (-\cos(\pi\gamma) \text{Ai}_{-1-\gamma} - \sin(\pi\gamma) \text{Gi}_{-1-\gamma}) \\ [\text{Gi}_\gamma, \text{Hi}_\gamma] &= \frac{\Gamma(1+\gamma)}{\pi} (\sin(\pi\gamma) \text{Ai}_{-1-\gamma} - \cos(\pi\gamma) \text{Gi}_{-1-\gamma} + \text{Hi}_{-1-\gamma}). \end{aligned}$$

## 5. GREEN'S FUNCTIONS

**5.1. The Green's function for (59).** We consider the linear problem

$$(59) \quad L_\gamma u := (1+\gamma)u + xu_x - u_{xxx} = f.$$

The identities of Proposition 4 and (52) imply explicit formulas for Greens functions in terms of generalized Airy and Scorer's functions. There is a unique right inverse

$T_L$  with integral kernel  $K_L(x, y)$  supported on the left of the diagonal. It is for  $x \geq y$

$$\begin{aligned}
 \frac{K_\gamma^L(x, y)}{\pi} &= \frac{\pi}{\Gamma(\gamma + 1)} \left\{ [\text{Ai}_\gamma, \text{Gi}_\gamma](y) \text{Hi}_\gamma(x) \right. \\
 &\quad \left. + [\text{Gi}_\gamma, \text{Hi}_\gamma](y) \text{Ai}_\gamma(x) + [\text{Hi}_\gamma, \text{Ai}_\gamma](y) \text{Gi}_\gamma(x) \right\} \\
 (60) \quad &= \text{Hi}_{-1-\gamma}(y) \text{Ai}_\gamma(x) + \text{Ai}_{-1-\gamma}(y) \text{Hi}_\gamma(x) \\
 &\quad + \sin(\gamma\pi) \left( \text{Gi}_{-1-\gamma}(y) \text{Gi}_\gamma(x) + \text{Ai}_{-1-\gamma}(y) \text{Ai}_\gamma(x) \right) \\
 &\quad + \cos(\gamma\pi) \left( \text{Ai}_{-1-\gamma}(y) \text{Gi}_\gamma(x) - \text{Gi}_{-1-\gamma}(y) \text{Ai}_\gamma(x) \right).
 \end{aligned}$$

It is easy to read off the leading terms of  $K_\gamma^L$  in various asymptotic regimes. Let  $x, y \gg 1$ . The leading term of the second line is given by the product of the Gi functions. It is

$$(61) \quad \sin(\pi\gamma) |x|^{-1-\gamma} |y|^\gamma.$$

The third line decays fast as  $x \sim y \rightarrow \infty$ .

For  $x, y \ll 0$  the only polynomial term without oscillations comes from the second line. It is

$$(62) \quad -\sin \pi\gamma \cos^2(\pi\gamma) |x|^{-1-\gamma} |y|^\gamma,$$

We recall that we will set  $\gamma = \frac{2}{p} - 1$  when we construct selfsimilar solutions, and we will search solutions of finite energy, i.e. with  $u_x \in L^2$  and  $u \in L^{p+2}$ . Let  $X_0 \subset C^1$  be the Banach space of functions such that the norm

$$(63) \quad \|u\|_{X_0} = \sup |(1 + |x|)^{1+\gamma} u| + |(1 + |x|)^{2+\gamma} u_x|$$

The decay of the generalized Airy functions and of Scorer's function determine uniquely the right inverse which maps compactly supported functions to  $X_0$ .

**Theorem 5.** *Let  $-1 < \gamma < 0$ . Then there exists a unique right inverse  $T_\gamma^a : C_0(\mathbb{R}) \rightarrow X$  with the integral kernel*

$$\begin{aligned}
 K_\gamma(x, y) &= \pi \left( \text{Hi}_{-1-\gamma}(y) \text{Ai}_\gamma(x) \chi_{y < x} - \text{Ai}_{-1-\gamma}(y) \text{Hi}_\gamma(x) \chi_{x < y} \right) \\
 (64) \quad &+ \pi \sin(\gamma\pi) \left( \text{Gi}_{-1-\gamma}(y) \text{Gi}_\gamma(x) + \text{Ai}_{-1-\gamma}(y) \text{Ai}_\gamma(x) \right) \chi_{x > y} \\
 &+ \pi \cos(\gamma\pi) \left( \text{Ai}_{-1-\gamma}(y) \text{Gi}_\gamma(x) - \text{Gi}_{-1-\gamma}(y) \text{Ai}_\gamma(x) \right) \chi_{x > y}
 \end{aligned}$$

Moreover  $T_\gamma$  maps  $C_0$  to  $X_0$ .

**5.2. The change of coordinates.** We will use the Green's function for the transformed problem. The equations

$$(1 + \gamma)v + yv_y - v_{yyy} = f$$

and

$$a((1 + \gamma)u + xu_x) - u_{xxx} + u_x = g$$

are equivalent via

$$(65) \quad x = a^{-1/3}y - a^{-1}, \quad v(y) = au(a^{-1/3}y - a^{-1}) \quad f(y) = g(x),$$

Then,

$$\begin{aligned}
u(x) &= a^{-1}v(a^{1/3}(x + a^{-1})) \\
&= a^{-1} \int K_\gamma(a^{1/3}(x + a^{-1}), z)f(z)dz \\
&= a^{-1} \int K_\gamma(a^{1/3}(x + a^{-1}), z)g(a^{-1/3}z - a^{-1})dz \\
&= a^{-2/3} \int K_\gamma(a^{1/3}(x + a^{-1}), a^{1/3}(y + a^{-1}))g(y) dy
\end{aligned}$$

Thus

$$u(x) = \int \tilde{K}_\gamma^a(x, y)g(y)dy$$

where

$$\tilde{K}^a(x, y) = a^{-2/3}K_\gamma(a^{1/3}(x + a^{-1}), a^{1/3}(y + a^{-1}))$$

We apply it to  $g = \partial_x F$ , where one integration by parts yields

$$(66) \quad u(x) = - \int \partial_y \tilde{K}^a(x, y)F(y)dy = \int K^a(x, y)F(y)dy =: T_\gamma^a F$$

with a new kernel

$$\begin{aligned}
(67) \quad \frac{a^{1/3}}{\pi} K^a(x, y) &= -\text{Ai}_{-\gamma}(a^{1/3}(y + a^{-1})) \text{Hi}_\gamma(a^{1/3}(x + a^{-1}))\chi_{x < y} \\
&\quad - \text{Hi}_{-\gamma}(a^{1/3}(y + a^{-1})) \text{Ai}_\gamma(a^{1/3}(x + a^{-1}))\chi_{y < x} \\
&\quad + \sin(\gamma\pi) \left( \text{Gi}_{-\gamma}(a^{1/3}(y + a^{-1})) \text{Gi}_\gamma(a^{-1/3}(x + a^{-1})) \right. \\
&\quad \left. + \text{Ai}_{-\gamma}(a^{1/3}(y + a^{-1})) \text{Ai}_\gamma(a^{1/3}(y + a^{-1})) \right) \chi_{x > y} \\
&\quad + \cos(\gamma\pi) \left( \text{Ai}_{-\gamma}(a^{1/3}(y + a^{-1})) \text{Gi}_\gamma(a^{-1/3}(x + a^{-1})) \right. \\
&\quad \left. - \text{Gi}_{-\gamma}(a^{1/3}(y + a^{-1})) \text{Ai}_\gamma(a^{1/3}(y + a^{-1})) \right) \chi_{x > y}.
\end{aligned}$$

We arrive at the reformulation

$$(68) \quad u(x) + T_\gamma^a(|u|^p u - \langle u, Q_x \rangle Q_x) = 0$$

of the bifurcation problem (12).

**5.3. Dependence on  $a$  and  $\gamma$ .** The previous considerations show that

$$\begin{aligned}
(69) \quad \frac{a^{2/3}K_L(x, y)}{\pi} &= \text{Hi}_{-1-\gamma}(a^{-2/3}(1 + ay)) \text{Ai}_\gamma(a^{-2/3}(1 + ax)) \\
&\quad - \text{Ai}_{-1-\gamma}(a^{-2/3}(1 + ay)) \text{Hi}_\gamma(a^{-2/3}(1 + ax)) \\
&\quad + \sin(\gamma\pi) \left( \text{Gi}_{-1-\gamma}(a^{-2/3}(1 + ay)) \text{Gi}_\gamma(a^{-2/3}(1 + ax)) \right. \\
&\quad \left. + \text{Ai}_{-1-\gamma}(a^{-2/3}(1 + ay)) \text{Ai}_\gamma(a^{-2/3}(1 + ax)) \right) \\
&\quad + \cos(\gamma\pi) \left( \text{Ai}_{-1-\gamma}(a^{-2/3}(1 + ay)) \text{Gi}_\gamma(a^{-2/3}(1 + ax)) \right. \\
&\quad \left. - \text{Gi}_{-1-\gamma}(a^{-2/3}(1 + ay)) \text{Ai}_\gamma(a^{-2/3}(1 + ax)) \right).
\end{aligned}$$

is the forward Green's function. Given  $y$  it is a solution to the homogeneous differential equation with initial condition

$$u(y) = u'(y) = 0, \quad u''(0) = 1$$

It depends analytically on  $x, y, a \in \mathbb{R}$  and  $\gamma$  away from the diagonal  $x = y$ . We claim that  $K^a$  is smooth with respect to  $a, \gamma, x$  and  $y$ . To see this we have to show that

$$a^{-1/3} \text{Ai}_\gamma(a^{-2/3}(1+ax)) \text{Hi}_{-\gamma}(a^{-2/3}(1+ay))$$

is smooth in  $a$  and  $\gamma$ . It suffices to consider this at  $x = y = 0$ , since solutions to analytic ODEs are analytic. We claim that

$$a^{-1/3} \text{Ai}_\gamma(a^{-2/3}) \text{Hi}_{-\gamma}(a^{-2/3})$$

is smooth with respect to  $a \in \mathbb{R}$ . Analyticity with respect to  $\gamma$  follows from analyticity of  $\text{Ai}_\gamma$  and  $\text{Hi}_{-\gamma}$  for fixed  $a$ . Smoothness in  $a$  and even analyticity is obvious for  $a \neq 0$ . At  $a = 0$  smoothness follows from the asymptotics of  $\text{Ai}_\gamma$  and  $\text{Hi}_{-\gamma}$  in (44) and (51).

The following Lemma quantifies the dependence on  $a$  in a crucial region. It is an immediate consequence of the asymptotics of the Airy and Scorer's functions.

**Lemma 6.** *The following estimate*

$$\begin{aligned} \left| K^a(x, y) - \left( e^{-\frac{1}{2}|x-y|} + a\chi_{x>y}(1+ax)^{-1-\gamma}(1+ay)^{-1+\gamma} \right) \right| \\ \leq c_\delta \left( a^2 + |a|e^{-\frac{1}{2}|x-y|} \right) \end{aligned}$$

holds for  $|x|, |y| \leq a^{-1/2}$ .

*Proof.* First we observe

$$\begin{aligned} \left| a^{-1/3} \text{Gi}_\gamma(a^{-2/3}(1+ax)) \text{Gi}_{-\gamma}(a^{-2/3}(1+ay)) - a(1+ax)^{-1-\gamma}(1+ay)^{-1+\gamma} \right| \\ \leq a^3(1+ax)^{-3} \end{aligned}$$

The terms  $\text{Ai}_{-\gamma} \text{Gi}_\gamma$ ,  $\text{Gi}_{-\gamma} \text{Ai}_\gamma$  and  $\text{Ai}_{-\gamma} \text{Ai}_\gamma$  are much smaller. To be precise we assume  $x \leq y$  and estimate

$$\begin{aligned} \pi^{-1} a^{-1/3} \left| \text{Ai}_{-\gamma}(a^{-2/3}(1+ay)) \text{Hi}_\gamma(a^{-2/3}(1+ax)e^{y-x}) - \frac{1}{2} \right| \\ = (1+ax)^{-\frac{1}{4}+\frac{\gamma}{2}} (1+ay)^{-\frac{1}{4}-\frac{\gamma}{2}} \left| \frac{1}{2} e^{\frac{2}{3a}((1+ax)^{\frac{3}{2}} - (1+ay)^{\frac{3}{2}}) + (y-x)} - \frac{1}{2} \right| \\ \leq c \left| e^{\frac{a}{4}(1+ax)^{-\frac{1}{2}}(y-x)^2} - 1 \right| \\ \leq ca|x-y|^2 \end{aligned}$$

The case  $x \geq y$  is similar. □

## 6. THE IMPLICIT FUNCTION THEOREM

**6.1. The operator  $T_\gamma^a$  in weighted function spaces.** We rewrite the problem as a fixed point problem for the identity plus a compact map. Then the Fredholm alternative will allow us to apply the implicit function theorem. Things however are not as simple as they may appear from this description: The derivatives with respect to  $a$  and  $\mu$  are not bounded in this functional analytic setting. They have to be handled by different arguments in the next section.

The following result is the basic linear estimate for the operator  $T_\gamma^a$ . It is a weighted estimate with a weight tailored for the problem at hand. This is necessarily involved.

The asymptotics on the left is essentially given by

$$\text{Hi}_\gamma(a^{-2/3}(1+ax)) / \text{Hi}_\gamma(a^{-2/3})$$

and on the right by

$$\text{Gi}_\gamma(a^{-2/3}(1+ax))/\text{Gi}_\gamma(a^{-2/3}).$$

This decay is to a certain extent captured by the weights below.

**Proposition 7.** *Let  $k \geq 0$ . There exists  $c > 0$  such that the following is true. Let  $0 < a \leq 1/10$ ,  $|\gamma + \frac{1}{2}| < \frac{1}{8}$  and*

$$(70) \quad w^a(x) := \begin{cases} e^{-\frac{1}{3a}(1+a^{-2/3}|1+ax|)^{-\frac{3}{8}}} & \text{if } x \leq -a^{-1} \\ \exp(\frac{1}{3a}[(1+ax)^{3/2}-1]) & \text{if } -a^{-1} \leq x \leq 0 \\ (1+x)^k(1+ax)^{-1-\gamma-k} & \text{if } x \geq 0 \end{cases}$$

and

$$(71) \quad w_i^a(x) = \begin{cases} e^{-\frac{1}{2a}(1+a^{-2/3}|1+ax|)^{-\frac{3}{2}}} & \text{if } x \leq -a^{-1} \\ \exp(\frac{1}{2a}[(1+ax)^{\frac{3}{2}}-1]) & \text{if } -a^{-1} \leq x \leq 0 \\ (1+x)^k(1+ax)^{-1-\gamma-k} & \text{if } x \geq 0 \end{cases}$$

and

$$(72) \quad w^0 = \begin{cases} e^{-|x|/2} & \text{if } x < 0 \\ 1 & \text{otherwise} \end{cases} \quad w_i^0 = \begin{cases} e^{-3|x|/4} & \text{if } x < 0 \\ 1 & \text{otherwise} \end{cases}$$

Then

$$(73) \quad \sup_{x, 0 \leq a \leq \frac{1}{2}, |\frac{1}{2} + \gamma| \leq \frac{1}{8}} |T_\gamma^a f(x)|/w^a(x) \leq c \sup_x |f(x)|/w_i^a(x).$$

The complexity of the weight reflects the different asymptotic areas, and the proof consists in decomposing operator and domain in smaller pieces for which elementary estimates become possible. The proposition is an immediate consequence of

**Lemma 8.** *There exists  $c > 0$  independent of  $x$ ,  $a$  and  $\gamma$  such that*

$$\int |K_\gamma^a(x, y)| w_i^a(y) dy \leq c w^a(x)$$

for  $x \in \mathbb{R}$ ,  $0 \leq a \leq 1$  and  $|\frac{1}{2} + \gamma| \leq \frac{1}{8}$ .

*Proof.* We will restrict ourselves to  $k = 0$ , with marginal differences for positive  $k$ . We recall that

$$(74) \quad |\text{Ai}_\gamma(x)| + |\text{Gi}_\gamma(x)| + |\text{Hi}_\gamma(x)| \leq c(1+|x|)^{-\frac{3}{8}}$$

for  $x \leq 0$  and  $|\gamma + \frac{1}{2}| \leq \frac{1}{8}$ .

**Step 1:**  $x \geq 0$ . There are contributions from the integrals over  $(-\infty, -a^{-1})$ ,  $(-a^{-1}, 0)$ ,  $(0, x)$  and  $(x, \infty)$ . We deal with them in reverse order, and we begin  $a^{-1/3} \text{Ai}_{-\gamma}(a^{-2/3}(1+ay)) \text{Hi}_\gamma(a^{-2/3}(1+ax))$  - the kernel for  $y > x$ . Then

$$\begin{aligned} & a^{-1/3}(1+ax)^{1+\gamma} \text{Hi}_\gamma(a^{-2/3}(1+ax)) \int_x^\infty \text{Ai}_{-\gamma}(a^{-2/3}(1+ay))(1+ay)^{-1-\gamma} dy \\ & \lesssim \int_x^\infty e^{\frac{2}{3a}((1+ax)^{\frac{3}{2}} - (1+ay)^{\frac{3}{2}})} \left(\frac{1+ax}{1+ay}\right)^{1+\frac{\gamma}{2}} [(1+ax)(1+ay)]^{-\frac{1}{4}} dy \\ & \lesssim \int_x^\infty e^{(1+ax)^{1/2}(x-y)} \left(\frac{1+ax}{1+ay}\right)^{\frac{5}{4}+\frac{\gamma}{2}} (1+ax)^{-\frac{1}{2}} dy \\ & \lesssim C \end{aligned}$$

holds uniformly in  $x \geq 0$  if  $|\gamma + \frac{1}{2}| \leq \frac{1}{8}$ . Similarly

$$a^{-1/3} \text{Ai}_\gamma(a^{-2/3}(1+ax)) \int_0^x \text{Hi}_{-\gamma}(a^{-2/3}(1+ay)) \left(\frac{1+ax}{1+ay}\right)^{1+\gamma} dy \leq c.$$

in the same range.



Next we consider the contribution of the product of the functions  $\text{Gi}$ , using  $\text{Gi}_\gamma(a^{-2/3}(1+ax)) \sim a^{\frac{2}{3}(1+\gamma)}(1+ax)^{-1-\gamma}$  for  $x > 0$ :

$$\begin{aligned} & a^{-1/3} \int_0^x \left( \frac{1+ax}{1+ay} \right)^{1+\gamma} \text{Gi}_\gamma(a^{-2/3}(1+ax)) \text{Gi}_{-\gamma}(a^{-2/3}(1+ay)) dy \\ & \lesssim \int_0^x a(1+ay)^{-2} dy \\ & \lesssim 1. \end{aligned}$$

The products

$$a^{-1/3} \text{Ai}_{-\gamma}(a^{-2/3}(1+ay)) \text{Gi}_\gamma(a^{-2/3}(1+ax))$$

and

$$a^{-1/3} \text{Gi}_{-\gamma}(a^{-2/3}(1+ay)) \text{Ai}_\gamma(a^{-2/3}(1+ax))$$

are much smaller.

We observe that

$$\text{Ai}_\gamma(a^{-2/3}(1+ax)) \lesssim \text{Ai}_\gamma(a^{-2/3}) w^a(x),$$

$$\text{Gi}_\gamma(a^{-2/3}(1+ax)) \lesssim \text{Gi}_\gamma(a^{-2/3}) w^a(x)$$

for  $x \geq 0$  and it suffices to bound the contribution from  $y \leq 0$  at  $x = 0$  to get the same bound for all nonnegative  $x$ .

The estimates

$$\begin{aligned} & \int_{-\infty}^0 \text{Hi}_{-\gamma}(a^{-2/3}(1+ay)) w_i^a(y) dy \lesssim \text{Hi}_{-\gamma}(a^{-2/3}), \\ (75) \quad & e^{-\frac{1}{2a}} \int_{-\infty}^{-a^{-1}} (1+a^{-2/3}(1+ay))^{\frac{1}{16}-\frac{3}{2}} dy \lesssim e^{-\frac{1}{3a}} \end{aligned}$$

and

$$\int_{-a^{-1}}^0 (1+a^{-2/3}(1+ay))^{-1-\gamma} e^{\frac{1}{2a}[(1+a^{-2/3}(1+ay))^{\frac{3}{2}}-1]} dy \lesssim a^{\frac{2}{3}(1+\gamma)} \sim \text{Gi}_\gamma(a^{-2/3})$$

are straight forward. This completes the estimate for  $x > 0$ .

**Step 2:**  $x < 0$ . In view of the first substep above (with  $x = 0$ ) the contribution from  $y > 0$  is controlled by the obvious estimate

$$\text{Hi}_\gamma(a^{-2/3}(1+ax)) \lesssim H_\gamma(a^{-2/3}) w^a(x).$$

We consider the contribution from  $y \leq 0$  to  $x \in [-a^{-1}, 0]$ . There are contributions from three different intervals:  $(-\infty, -a^{-1})$ ,  $(-a^{-1}, x)$ , and  $(x, 0)$ , which we consider step by step. We consider first the product of  $\text{Ai}$  and  $\text{Hi}$ . The desired estimate is

$$\begin{aligned} & \sup_{-a^{-1} \leq x \leq 0} a^{-1/3} \int_{-a^{-1}}^0 \frac{(1+a^{-2/3}(1+ay))^{\gamma-1}}{(1+a^{-2/3}(1+ax))^{\gamma+1}} \times \\ & \times e^{-\frac{2}{3a}[(1+ax)^{\frac{3}{2}}-(1+ay)^{\frac{3}{2}}]} e^{-\frac{1}{2a}[(1+ay)^{\frac{3}{2}}-1]} e^{\frac{1}{3a}[(1-ax)^{\frac{3}{2}}-1]} dy \end{aligned}$$

which is trivial once broken up into different cases:  $x = 0$ ,  $y \leq x$ ,  $-\frac{1}{2}a^{-1} \leq x < y$  and  $-a^{-1} \leq x \leq -\frac{1}{2a}$ . The contributions from the other terms in the Greens function are much smaller. Finally the contribution (to  $x \in [-a^{-1}, 0]$ ) from  $y \leq -a^{-1}$  is controlled by (75).

**Step 3: The case  $x < -a^{-1}$ , contribution from  $y \leq 0$ .** Again we have to consider the integrals over  $(-\infty, x)$ ,  $(x, a^{-1})$  and  $(a^{-1}, 0)$ . The integral over  $(a^{-1}, 0)$  has been evaluated above. The obvious estimates

$$|\text{Ai}_\gamma(a^{-2/3}(1+ax))| + |\text{Gi}_\gamma(a^{-2/3}(1+ax))| + \text{Hi}_\gamma(a^{-2/3}(1+ax)) \lesssim \frac{w^a(x)}{w^a(-a^{-1})}$$

complete that part.

The kernel satisfies

$$|K_\gamma^a(x, y)| \lesssim a^{-1/3} (1 + a^{-2/3}(1 + ax))^{-\frac{3}{8}} (1 + a^{-2/3}(1 + ay))^{-\frac{1}{16}}$$

for  $x, y \leq -a^{-1}$ . Now

$$\int_{-\infty}^{-a^{-1}} (1 + a^{-2/3}(1 + ay))^{-\frac{2}{3} + \frac{1}{8}} dy \lesssim a^{-1/3}$$

completes the proof.  $\square$

We reformulate the bifurcation problem as a fixed point problem

$$(76) \quad u(x) = T_\gamma^a (|u|^p u - \langle u, Q_x \rangle Q_x)$$

where we search  $u$  in a neighborhood of  $Q$ .

We introduce  $v = u/w^a$  with  $w^a$  from (70) and rewrite the problem as

$$F(v) = 0$$

with

$$F(v) = v - (w^a)^{-1} T_\gamma^a [|vw^a|^p vw^a - \langle vw^a, Q_x \rangle Q_x].$$

Proposition 7 implies that the map is  $j$  times Frechet differentiable on the space of bounded continuous functions, for every nonnegative integer  $j \leq 3$ .

We turn to the study of the linearization of (76) at  $Q$ . We include the weights into the operator and consider

$$\tilde{T}(a, p, \gamma, v)u := (w^a)^{-1} T_\gamma^a [(p+1)|vw^a|^p w^a u + \langle u, w^a Q_x \rangle Q_x].$$

Let  $C_b(\mathbb{R})$  denote the space of continuous functions equipped by the supremum norm, and the closed subspace of functions with limit 0 as  $x \rightarrow \pm\infty$  by  $C_0$ . The space of linear operators from the normed space  $X$  to the normed space  $Y$  is denoted by  $L(X, Y)$ , which we equip with the operator norm.

**Corollary 9.** *The map*

$$\begin{aligned} \left( [0, 1] \times \left[ -\frac{1}{2} - \frac{1}{8}, -\frac{1}{2} + \frac{1}{8} \right] \times [3, q] \times C_0(\mathbb{R}) \right) &\rightarrow L(C_b, C_b) \\ (a, \gamma, p, v) &\rightarrow \left( u \rightarrow \tilde{T}(a, p, \gamma, v)u \right) \end{aligned}$$

*is continuous.*

*Proof.* The map

$$(a, p) \rightarrow Q_x/w_i^a \in L^1$$

is clearly continuous as is

$$(v, u, a, p) \rightarrow |vw^a|^p w^a u/w_i^a.$$

This implies continuity with respect to  $v$ , uniform with respect to  $a$  and  $p$ . Hence it suffices to prove the continuity for the composition with the multiplication by a characteristic function,

$$(a, \gamma, p, v) \rightarrow \left( u \rightarrow \tilde{T}(a, p, \gamma, v) \chi_{[-R, R]} u \right).$$

The proof of Proposition 7 implies that

$$x \rightarrow \tilde{T}(a, p, \gamma, v) \chi_{[-R, R]} u(x)$$

converges to zero as  $x \rightarrow -\infty$ , uniformly for bounded  $v$  and  $u$  and  $a$  and  $p$  as in the theorem. Continuity with respect to  $p$  is obvious. On the right hand side the situation is slightly different: Since we apply the operator to a function with compact support, the only terms which does not decay as  $x \rightarrow \infty$  comes from  $\text{Gi}_\gamma(a^{-2/3}(1 + ax))$ . This term is clearly continuous with respect to  $a$  and  $\gamma$ . Continuity with

respect to  $\gamma$  and  $a$  follows from the continuity of the Airy and Scorer functions, their asymptotics and the continuity of the Green's function.  $\square$

The invertibility of the linearization in a neighborhood of the bifurcation point is contained in the next proposition. We denote

$$S_v^a u = u - w_a^{-1} T_\gamma^a [(p+1)w_a^{p+1} v^p u - \langle u, w_a Q_x \rangle Q_x]$$

**Proposition 10.** *There exists  $\delta > 0$  such that  $S_v^a : C_b \rightarrow C_b$  is invertible with an inverse whose norm is uniformly bounded for*

$$|\gamma + \frac{1}{2}| \leq \delta, 0 \leq a \leq \delta \text{ and } \sup \frac{|v - Q|}{w^a} \leq \delta.$$

*Proof.* The operator  $S_v^a$  is bounded by Proposition 7 and the norm continuity at  $v = Q/w^a$  is the content of Corollary 9. It thus suffices to consider invertibility at  $a = 0$  and  $v = Q/w^a$ . Clearly

$$u \rightarrow (w^0)^{-1} T^0 [(p+1)Q^p (w^0)^{p+1} u] - \langle u, w^0 Q_x \rangle Q_x$$

is compact. We recall that the integral kernel of  $T^0$  is  $\frac{1}{2}e^{-|x-y|}$ .

By the Fredholm alternative  $S_Q^0$  is invertible if the null space is trivial. Consider

$$(77) \quad u = T_\gamma^0 [(p+1)Q^p u - \langle u, Q_x \rangle Q_x].$$

We claim that there is only the trivial bounded solution. Suppose that  $u$  satisfies the homogeneous equation (77). Since the kernel decays fast also  $u$  decays fast, and the same holds for the derivatives. Hence

$$u - u_{xx} - (p+1)Q^p u + \langle u, Q_x \rangle Q_x = Lu - \langle u, Q_x \rangle Q_x = 0.$$

We take the inner product with  $Q_x$ . Then  $\langle u, Q_x \rangle = 0$  since  $LQ_x = 0$ . The null space of  $L$  is spanned by  $Q_x$  and hence  $u = 0$ . This null space is trivial, by the Fredholm alternative  $S_Q^0$  is invertible, and this remains so in a small neighborhood of the coefficients and  $Q/w^a$ .  $\square$

We continue with an estimate which implies that  $Q$  is almost a solution to the fixed point problem. This is important since  $F$  fails to be differentiable with respect to  $a$  and  $\gamma$ .

**Lemma 11.** *There exists  $C > 0$  such that*

$$\sup_x |(w^a)^{-1} (Q - T_\gamma^a Q^{p+1})| \leq Ca.$$

*Proof.* We observe that

$$Q - T_\gamma^0 [Q^{p+1} + \langle Q, Q_x \rangle Q_x] = 0$$

since  $Q$  satisfies the soliton equation. The assertion is equivalent to

$$|(T_\gamma^0 - T_\gamma^a)Q^{p+1}| \leq caw^a(x)$$

which we address now. Since  $Q \lesssim e^{-|x|}$  there exists  $c > 0$  independent of  $a$ ,  $\gamma$  and  $p$

$$\sup_{|x| \geq c|\ln a|} |(w_i^a)^{-1} Q^{p+1}| \leq a.$$

so that with  $\chi$  the characteristic function of the complement of  $[-c|\ln a|, c|\ln a|]$

$$\sup_x (w^a(x))^{-1} |T_\gamma^a \chi Q^{p+1}(x)| \lesssim a.$$

it suffices to verify

$$|(T_\gamma^0 - T_\gamma^a)\chi Q^{p+1}| \lesssim aw^a(x).$$

Checking the kernel we see that

$$|T_\gamma^0 \tilde{\chi} Q^{p+1}(x)| + |T_\gamma^a \tilde{\chi} Q^{p+1}(x)| \lesssim a w^a(x)$$

if  $|x| \geq 2|\ln(a)|$ . Now  $|Q^p| \leq e^{-p|x|}$ ,  $Q^s$  is integrable whenever  $s > 0$  and  $Q \lesssim w_i^a$ . Thus the statement will follow from

$$(78) \quad \sup_{|x|, |y| \leq c|\ln a|} e^{\frac{2}{3} \max\{-x, 0\}} \left( K_\gamma^a(x, y) - \frac{1}{2} e^{-|x-y|} \right) e^{-3|y|} \lesssim a$$

which is a consequence of Lemma 6.  $\square$

**Proposition 12.** *Let  $q > 4$  and  $3 \leq p \leq q$ . Then there exists  $\varepsilon$  and  $C > 0$  so that there is a unique fixed point  $u$  to*

$$u = T_\gamma^a(|u|^p u - \langle u, Q_x \rangle Q_x)$$

with

$$(79) \quad \sup_x (w^a(x))^{-1} |u(x) - Q(x)| + |\langle u, Q_x \rangle| \lesssim a$$

for

$$\max \left\{ \left| \frac{1}{2} + \gamma \right|, a \right\} \leq \varepsilon.$$

The map  $(a, \gamma, p) \rightarrow (w^a)^{-1} u \in C_b(\mathbb{R})$  is continuous.

*Proof.* We write  $u = w^a v - Q$ . Then we search a fixed point to

$$\begin{aligned} v &= (w^a)^{-1} (T_\gamma^a(|w^a v + Q|^p (w^a v + Q) - \langle v, Q_x \rangle Q_x) - Q) \\ &= (w^a)^{-1} (T_\gamma^a(|w^a v + Q|^p (w^a v + Q) - Q^{p+1} - \langle v, Q_x \rangle Q_x)) \\ &\quad + (w^a)^{-1} (T_\gamma^a Q^{p+1} - Q). \end{aligned}$$

The second term on the right hand side is bounded by a constant times  $a$  by Lemma 11. The derivative at  $v = 0$  is invertible by Lemma 10 with a uniformly bounded inverse. The existence of a unique fixed point with the desired properties follows now by the implicit function theorem.  $\square$

## 7. ASYMPTOTICS AND DIFFERENTIABILITY

In the last section we have constructed a unique fixed point  $u$  to

$$(80) \quad u = T_\gamma^a(|u|^p u - \langle u, Q_x \rangle Q_x)$$

with

$$\|(u - Q)/w^a\|_{sup} < 1.$$

Moreover it satisfies

$$(81) \quad |u - Q| \leq c a w^a$$

with a constant which is uniform in  $a$ ,  $p$  and  $\gamma$ . It follows immediately from the integral representation and the decay that  $u_x$  is square integrable. Moreover  $u/w^a$  depends continuously on  $a$ ,  $p$  and  $\gamma$  considered as a map to  $C_b(\mathbb{R})$ . It remains to show that this map is smooth for every  $x$ , to give bounds for the derivatives, and to prove the uniqueness statement. Here we turn to differentiability and bounds for the derivatives.

As a first step and a warm up we consider the simpler term first. This term will not enter the asymptotics of the fixed point, but we need it to prove differentiability with respect to  $a$ .

7.1. **The asymptotics of  $v_\gamma^a := T_\gamma^a Q_x$ .** We define

$$(82) \quad c_0 = c_0(a, p, \gamma) = \pi \int \text{Ai}_{1-\gamma}(a^{-2/3}(1+ay))Q(y)dy$$

and

$$(83) \quad d_0 = d_0(a, p, \gamma) = \pi \int \text{Gi}_{1-\gamma}(a^{-2/3}(1+ay))Q(y)dy.$$

**Proposition 13.** *The following estimates hold for  $\kappa < 1$*

$$(84) \quad e^{-\kappa x} \left| \partial_x^j \partial_a^k \partial_p^l (T_\gamma^a Q_x + c_0 \text{Hi}_\gamma(a^{-2/3}(1+ax))) \right| \leq \frac{c(j, k, l)}{\sqrt{1-\kappa}}$$

if  $x \leq 0$  and if  $x \geq 0$

$$(85) \quad e^{\kappa x} \left| \partial_x^j \partial_a^k \partial_p^l (T_\gamma^a Q_x + d_0 \text{Gi}_\gamma(a^{-2/3}(1+ax))) \right| \leq \frac{c(j, k, l)}{\sqrt{1-\kappa}}.$$

Moreover there are the asymptotic series

$$c_0 H_\gamma(a^{-2/3}) = \sum_{k=0}^{\infty} \alpha_k(\gamma, p) a^k$$

and

$$d_0 \text{Gi}_\gamma(a^{-2/3}) = \sum_{k=2}^{\infty} \beta_k(\gamma, p) a^k$$

with nontrivial leading term  $\beta_2$  resp  $\alpha_0$ . The coefficients are smooth functions of  $\gamma$  and  $p$ , with bounds depending only on  $k$ .

*Proof.* We can define a solution to the linear equation

$$(86) \quad a(\gamma v + x v_x) - v_{xx} + v_x = Q_{xx}$$

by an integral kernel  $K_L$ , supported in  $y < x$ , which is given by (compare with Theorem 5)

$$(87) \quad \begin{aligned} \frac{a^{2/3} K_L(x, y)}{\pi} = & \text{Hi}_{-1-\gamma}(a^{-2/3}(1+ay)) \text{Ai}_\gamma(a^{-2/3}(1+ax)) \\ & + \text{Ai}_{-1-\gamma}(a^{-2/3}(1+ay)) \text{Hi}_\gamma(a^{-2/3}(1+ax)) \\ & + \sin(\gamma\pi) \left( \text{Gi}_{-1-\gamma}(a^{-2/3}(1+ay)) \text{Gi}_\gamma(a^{-2/3}(1+ax)) \right. \\ & \quad \left. + \text{Ai}_{-1-\gamma}(a^{-2/3}(1+ay)) \text{Ai}_\gamma(a^{-2/3}(1+ax)) \right) \\ & + \cos(\gamma\pi) \left( \text{Ai}_{-1-\gamma}(a^{-2/3}(1+ay)) \text{Gi}_\gamma(a^{-2/3}(1+ax)) \right. \\ & \quad \left. - \text{Gi}_{-1-\gamma}(a^{-2/3}(1+ay)) \text{Ai}_\gamma(a^{-2/3}(1+ax)) \right). \end{aligned}$$

Two solutions to (86) differ by a solution to the homogeneous problem. The formula

$$\int_{-\infty}^x K_L(x, y) Q_{xx}(y) dy - c_0 \text{Hi}_\gamma(a^{-2/3}(1+ax))$$

defines a solution to (86) hence it differs from  $v_\gamma^a$  by a solution to the homogeneous equation. Both functions and their derivatives are bounded by a multiple of  $w^a$  for  $x \leq 0$ , and hence their difference is a multiple of  $\text{Hi}_\gamma$ . But the coefficients of the leading term are the same because of the choice of  $c_0$ , and hence both are the same.

We have

$$\partial_x^j \left[ v - c_0 \text{Hi}_\gamma(a^{-2/3}(1+ax)) \right] = \int_{-\infty}^x \partial_x^j K_L(x, y) Q_{xx}(y) dy$$

if  $j = 0, 1, 2$ . For  $j \geq 3$  there is an additional term consisting of a finite sum of  $Q$  and its derivatives. The kernel obviously reproduces exponential decay up to polynomial factors. The leading contribution for  $\kappa \rightarrow 1$  comes from  $x$  close to 1.

We argue similarly for  $x \geq 0$ , but this time with the standard kernel  $K_\gamma^a$ . The leading part comes from  $a^{-1/3} \text{Gi}_\gamma(a^{-2/3}(1+ax)) \text{Gi}_{-\gamma}(a^{-2/3}(1+ax))$ . The products of  $\text{Ai}_\gamma \text{Hi}_{-\gamma}$  and  $\text{Ai}_{-\gamma} \text{Hi}_\gamma$  and reproduce exponential decay if  $\kappa < 1$  and with a constant  $\sqrt{1-\mu}$  if  $\mu$  is close to 1. The other components of the kernel are supported in  $y \geq x$ . They reproduce the exponential decay of  $Q_{xx}$ . We verify the asymptotic formulas for  $c_0$  and  $d_0$  and we begin with  $d_0$ . Let

$$m_k = \int x^k Q(x) dx.$$

be the moments of  $Q$ . They are smooth functions of  $p$ , and independent of  $\gamma$  and  $a$ . A Taylor expansion of  $\text{Gi}_{1-\gamma}$  gives the asymptotic series

$$\begin{aligned} \int \text{Gi}_{1-\gamma}(a^{-2/3}(1+ay))Q(y)dy &\sim \sum_{k=0}^{\infty} \frac{1}{k!} m_k a^{k/3} \text{Gi}_{1+k-\gamma}(a^{-2/3}) \\ &\sim a^{2-\frac{2\gamma}{3}} (m_0 + (m_1 + m_0)a + \dots) \\ &\sim (\text{Gi}_\gamma(a^{-2/3}))^{-1} \left( \sum_{k=2}^{\infty} \beta_k a^k \right) \end{aligned}$$

with  $\beta_2 \neq 0$ .

Differentiability of the coefficients with respect to  $p$  is obvious. Differentiability with respect to  $\gamma$  follows from the differentiability of  $\text{Gi}_{1-\gamma}$  with respect to  $\gamma$  and the corresponding bounds. The difference to a partial sum is easily controlled by the estimates for  $\text{Gi}_\gamma$  and its derivatives.

For the expansion of  $c_0$  we write

$$\begin{aligned} c_0 &= a^{-2/3} \int \text{Ai}_{1-\gamma}(a^{-2/3}(1+ay))e^y(e^{-y}Q(y))dy \\ &= -a^{-2/3} \int \int_0^y \left[ \text{Ai}_{1-\gamma}(a^{-2/3}(1+as))e^s \right] ds \frac{d}{dy}(e^{-y}Q(y))dy. \end{aligned}$$

The function  $\frac{d}{dy}(e^{-y}Q(y))$  is a Schwartz function with exponential decay. The zeroth moment is  $-1$ , but  $\int_0^x \text{Ai}_{1-\gamma}$  vanishes at  $x = 0$ , and the leading contribution comes from the next term,

$$\begin{aligned} c_0 &= a^{-2/3} \text{Ai}_{1-\gamma}(a^{-2/3}) \int y \frac{d}{dy}(e^{-y}Q(y))dy \\ &\quad + \left( \frac{1}{4} + \frac{\gamma}{2} \right) (a^{1/3} \text{Ai}_{1-\gamma}(a^{-2/3})) \int y^2 \frac{d}{dy} e^{-y} Q(y) dy \dots \\ &= (\text{Hi}_\gamma(a^{-2/3}))^{-1} \left( \sum_{j=0}^{\infty} \alpha_j a^j \right) \end{aligned}$$

with  $\alpha_0 \neq 0$ . Any derivative on  $\text{Ai}_{1-\gamma}(a^{-2/3}(1+ax))e^x$  gains us a factor  $a$ . This is a consequence of the multiplication by  $e^x$ . We used the expansion of  $\text{Hi}_\gamma$  in this expansion. The derivatives with respect to  $p$  fall only on  $Q$ , and hence they are easy to estimate. The Scorer functions are differentiable with respect to  $\gamma$ . This implies the statement on the differentiability of the coefficients.  $\square$

As a consequence we have

$$|\langle u, Q_x \rangle (v_\gamma^a - c_0 \text{Hi}_\gamma(a^{-2/3}(1+ax)))| \leq cae^{\kappa x}$$

hence

$$|\langle u, Q_x \rangle v_\gamma^a| \leq ca \left( \frac{\text{Hi}_\gamma(a^{-\frac{2}{3}}(1+ax))}{\text{Hi}_\gamma(a^{-\frac{2}{3}})} + \frac{e^{\kappa x}}{\sqrt{1-\kappa}} \right)$$

for  $x \leq 0$  and similarly, for  $x > 0$ ,

$$(88) \quad |\langle u, Q_x \rangle v^a| \leq ca \left( a^2 \frac{\text{Gi}_\gamma(a^{-2/3}(1+ax))}{\text{Gi}_{-\gamma}(a^{-2/3})} + \frac{e^{-\kappa x}}{\sqrt{1-\kappa}} \right)$$

In the sequel we will only rely on those two estimates, and not on the full statement of Proposition 13.

**7.2. Bounds for the fixpoints and derivatives with respect to  $x$ .** After this warm-up we turn to the nonlinear term. Let

$$c_1 = a^{-1/3} \int \text{Ai}_{-\gamma}(a^{-2/3}(1+ay)) u^{p+1}(y) dy$$

and

$$d_1 = a^{-4/3} \int \text{Gi}_{-\gamma}(a^{-2/3}(1+ay)) u^{p+1}(y) dy.$$

These integrals exist since  $|u| \lesssim w^a(x)$ . Using the bound

$$|u - Q| \leq caw^a(x)$$

of the previous section we see that

$$|u^{p+1} - Q^{p+1}| \leq caw_i^a(x)$$

and, as in the previous subsection

$$c_1 = (1 + O(a)) a^{-1/3} \text{Ai}_{-\gamma}(a^{-2/3}) \int Q^{p+1}(y) dy.$$

Thus

$$(89) \quad c_1 \sim (\text{Hi}_\gamma(a^{-2/3})^{-1}).$$

and similarly

$$(90) \quad d_1 \sim a^{-\frac{2(1+\gamma)}{3}}.$$

The function  $u^{p+1}$  decays sufficiently fast to repeat the argument of the last section. Thus

$$w_L(x) := u(x) - v^a - c_1 \text{Hi}_\gamma(a^{-2/3}(1+ax)) = \int_{-\infty}^x K_L^a(x, y) \partial_y u^{p+1}(y) dy$$

and we also have the obvious integral representation for

$$w_R(x) := u(x) - v^a - d_1 \text{Gi}_\gamma.$$

Thus we obtain the very rough estimate, using  $p \geq 3$  and  $\gamma$  close to  $-1/2$

$$|w_L(x)| \leq c^{p+1} \int_{-\infty}^x |\partial_y K_L(x, y)| (w^a(x))^{p+1} dx \leq c \left( \frac{\text{Hi}_\gamma(a^{-2/3}(1+ax))}{\text{Hi}_\gamma(a^{-2/3})} \right)^4.$$

We put this information in the expansion. The oscillatory part of the kernel gives a small contribution when applied to  $|x|^{-(\gamma+1)(p+1)}$  resp.  $(\text{Hi}_\gamma(a^{-2/2}(1+ax)))^{p+1}$  hence, with  $\omega = -(1+\gamma)(p+1) - p - 2$

$$(91) \quad |w_L(x)| \lesssim \begin{cases} a^{-1/3} |1 + a^{-2/3}(1+ax)|^\omega (\text{Hi}_\gamma(a^{-2/3}))^{-p-1} & \text{if } x \leq -a^{-1} \\ |a^{2/3} + (1+ax)|^{-1/2} \left( \frac{\text{Hi}_\gamma(a^{-2/3}(1+ax))}{\text{Hi}_\gamma(a^{-2/3})} \right)^{p+1} & \text{if } -a^{-1} \leq x \leq 0. \end{cases}$$

Similarly we repeat the arguments from the last section on the right hand side. In a first step

$$|w_R(x)| \leq c \left( e^{-x} + a \frac{\text{Gi}_\gamma(a^{-2/3}(1+ax))}{\text{Gi}_\gamma(a^{-2/3})} \right).$$

We plug this into the integral operator. The exponential part with Ai and Hi reproduces the decay  $(Q + ca(1+ax)^{-1-\gamma})^{p+1}$ . The second potentially large contribution is bounded by

$$\begin{aligned} a \int_x^\infty (1+ay)^{-1+\gamma} (Q + ca(1+ax)^{-1-\gamma})^{-(p+1)} dy (1+ax)^{-1-\gamma} \\ \lesssim ae^{-x} + a^{p+1} (1+ax)^{-(p+1)\gamma-(p+2)} \end{aligned}$$

The exponential part again reproduces the decay and we arrive at

$$|w_R(x)| \leq ce^{-x} + a^{p+1} (1+ax)^{-(1+\gamma)(1+p)-1}$$

for  $x > 0$ . All three expansions remain correct under differentiation. We collect the estimates in the following lemma.

**Lemma 14.** *There exists  $\varepsilon > 0$  so that for*

$$0 \leq a \leq \varepsilon, \quad \left| \frac{1}{2} + \gamma \right| \leq \varepsilon, \quad 3 \leq p \leq q$$

*and the fixed point  $u$  the following is true. Let*

$$w_L = u - v^a - c_1 \text{Hi}_\gamma(a^{-2/3}(1+ax)) - Q.$$

*Then*

$$|\partial_x^k w_L| \leq c \begin{cases} \frac{a^{-1/3} |1 + a^{-2/3}(1+ax)|^{-(p+1)\gamma-p-2}}{(\text{Hi}_\gamma(a^{-2/3}))^{p+1}} & \text{if } x \leq -a^{-1} \\ (a^{2/3} + (1+ax))^{\frac{k-1}{2}} \left( \frac{\text{Hi}_\gamma(a^{-2/3}(1+ax))}{\text{Hi}_\gamma(a^{-2/3})} \right)^{p+1} & \text{if } -a^{-1} \leq x \leq 0 \end{cases}.$$

*and with*

$$w_R = u - v^a - ad_1 \text{Gi}_\gamma(a^{-2/3}(1+ax)) - Q$$

*the estimate*

$$|\partial_x^k w_R| \leq c_k a \left[ e^{-x} + a^{p+k} (1+ax)^{-(p+1)\gamma-p-2-k} \right]$$

*holds. The sum  $(c_0 + c_1)$  and  $d_1$  are bounded and bounded from below by a positive constant, independent of  $p$ ,  $\gamma$  and  $a$ . Finally*

$$(92) \quad c^{-1} \frac{\text{Hi}_\gamma(a^{-2/3}(1+ax))}{\text{Hi}_\gamma(a^{-2/3})} \leq u \leq c \frac{\text{Hi}_\gamma(a^{-2/3}(1+ax))}{\text{Hi}_\gamma(a^{-2/3})}$$

*if  $x \leq 0$  and*

$$(93) \quad c^{-1}(e^{-x} + a(1+ax)^{-1-\gamma}) \leq u \leq c(e^{-x/2} + a(1+ax)^{-1-\gamma})$$

*if  $x \geq 0$ .*

*Proof.* Only the last two statements need to be shown. Since

$$|u - Q| \leq aw^a$$

the statement is obvious for  $|x| \leq |\ln a|/2$ .

For  $x \leq -|\ln a| + R$  and  $a$  sufficiently small the term  $\frac{\text{Hi}_\gamma(a^{-2/3}(1+ax))}{\text{Hi}_\gamma(a^{-2/3})}$  becomes dominant and ensures positivity for those  $x$ . The same argument applies on the right hand side.  $\square$

In particular  $u$  is positive and bounded from below by the same type of bounds as from above.



**7.3. Derivatives with respect to  $\gamma$  and  $a$ .** The result of this subsection concludes the proof.

**Proposition 15.** *The fixed point  $u$  is infinitely often differentiable with respect to  $x$ ,  $a$ ,  $\gamma$  and  $p$  up to  $a = 0$ . Moreover the estimate*

$$|\partial_x^k \partial_a^l \partial_p^m \partial_\gamma^n u| \lesssim \begin{cases} (a^{-2/3} + |1 + ax|)^{-1-\gamma-k-n} |\ln(2 + |ax|)|^n / \text{Hi}_\gamma(a^{-2/3}) & \text{if } x < -a^{-1} \\ \text{Hi}_\gamma(a^{-2/3}(1 + ax)) / \text{Hi}_\gamma(a^{-2/3}) & \text{if } -a^{-1} \leq x \leq 0 \\ e^{-x} + a^{1+k}(1 + ax)^{-1-\gamma-k-n} |\ln(2 + ax)|^n & \text{if } x \geq 0 \end{cases}$$

holds with a constant depending only on  $k, l, m$  and  $n$ .

The bounds are exactly the bounds for the derivatives of

$$a(1 + ax)^{-1-\gamma}$$

plus a Schwartz function, resp. for  $x \leq 0$

$$\text{Hi}_\gamma(a^{-2/3}(1 + ax)) / \text{Hi}_\gamma(a^{-2/3})$$

Proposition 15 completes the proof of Theorem 1.

Despite considering a nonsmooth nonlinearity the fixed point will be smooth. This is compatible with the nonregularity of the power function since the fixed point  $u$  is positive.

*Proof.* The differentiation with respect to  $p$  is simpler than the differentiation with respect to  $a$  and  $\gamma$ , and we ignore it. We differentiate

$$a((1 + \gamma)u + xu_x) - u_{xxx} + u_x + \partial_x(|u|^p u + \langle u, Q_x \rangle Q_x) = 0$$

with respect to  $\gamma$  formally and denote the derivative again by  $\dot{u}$ . It satisfies

$$a((1 + \gamma)\dot{u} + x\dot{u}_x) - \dot{u}_{xxx} + \dot{u}_x + \partial_x((p + 1)|u|^p \dot{u} + \langle \dot{u}, Q_x \rangle Q_x) = -au$$

By Proposition 10 the linear operator is invertible, and we want estimate  $\dot{u}$  in terms of  $u$ . However, we do not have the bound  $|u| \leq w_i^a$  for  $|x| \leq -a^{-1}$  since there  $w_i^a$  is not bounded by  $w^a$ .

We choose a smooth monotone function  $\eta_+$ , supported in  $[-1, \infty)$  and identically one in  $[1, \infty)$ . Let  $\eta(x) = 1 - \eta_+(x)$ . We denote

$$\dot{\text{Hi}}_\gamma = \frac{\partial}{\partial \gamma} \text{Hi}_\gamma$$

and

$$\dot{\text{Gi}}_\gamma = \frac{\partial}{\partial \gamma} \text{Gi}_\gamma$$

Then

$$a((1 + \gamma)\dot{\text{Hi}}_\gamma + x\dot{\text{Hi}}_\gamma') - \dot{\text{Hi}}_\gamma''' + \dot{\text{Hi}}_\gamma' = -a \text{Hi}_\gamma$$

Let

$$\dot{v} = \dot{u} - (c_0 + c_1)\eta_- \dot{\text{Hi}}_\gamma - a(d_0 + d_1)\eta_+ \dot{\text{Gi}}_\gamma$$

Then

$$\begin{aligned} & a(1 + \gamma)\dot{v} + x\dot{v}_x) - \dot{v}_{xxx} + \dot{v}_x + \partial_x((p + 1)|u|^p \dot{v} + \langle \dot{v}, Q_x \rangle Q_x) \\ &= \phi - a(u - (c_0 + c_1)\eta_- \text{Hi}_\gamma - a(d_0 + d_1)\eta_+ \text{Gi}_\gamma) \\ & \quad - \partial_x((p + 1)|u|^p [(c_0 + c_1)\eta_- \dot{\text{Hi}}_\gamma + a(d_0 + d_1)\eta_+ \dot{\text{Gi}}_\gamma]) \\ & \quad + \langle (c_0 + c_1)\eta_- \dot{\text{Hi}}_\gamma + a(d_0 + d_1)\eta_+ \dot{\text{Gi}}_\gamma, Q_x \rangle Q_x. \end{aligned}$$

for some smooth function  $\phi$  supported in  $[-1, 1]$ . The right hand side decays sufficiently fast to apply Proposition 10. To justify this formal argument we use finite

differences. Continuity with respect to all parameters is obvious. This argument can be iterated.

Similarly we deal with derivatives with respect to  $a$ . The partial derivatives  $\partial_a^n \text{Gi}_\gamma(a^{-1/2}(1+ax))$  behave similarly as

$$\partial_a^n (1+ax)^{-1-\gamma} = c_{\gamma,n} \frac{x^n}{(1+ax)^{-n-\gamma}}.$$

Again Proposition 10 implies differentiability with respect to  $a$ , for  $a > 0$ , but this time we have to use weights with  $k > 0$ .

There is basically no difference in applying this argument to the derivative with respect to  $a$ ,  $p$  or  $x$ , using crucially the estimate (92) and (93).  $\square$

**7.4. Expansion of the selfsimilar solution.** The argument above gives information on the asymptotics of the self similar solutions which we state below.

**Proposition 16.** *Let  $u = u(a)$  be the selfsimilar solution orthogonal to  $Q_x$ . Then exists a unique expansion*

$$\left| u(x) - a(1+ax)^{-2/p} \sum_{j=0}^N d_j(a)(1+ax)^{-3j} \right| \leq ac_N(1+ax)^{-3N-3}$$

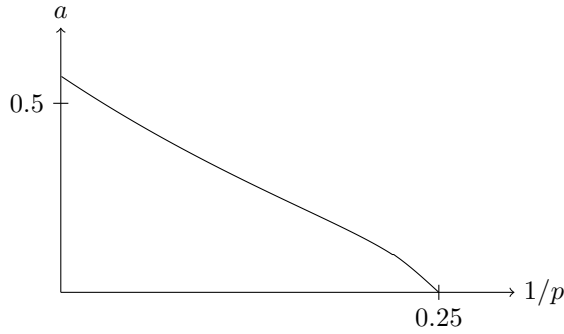
for  $x > 1$ , where  $d_j$  are bounded uniformly in  $a$ , and  $c_N$  is independent of  $a$  and

$$\left| u(x) - (\text{Hi}_\gamma(a^{-2/3}))^{-1} |1+ax|^{-2/p} \sum_{j=0}^N d_j(a)(1+ax)^{-3j} \right| \leq ac_N(1+|ax|)^{-3N-3}$$

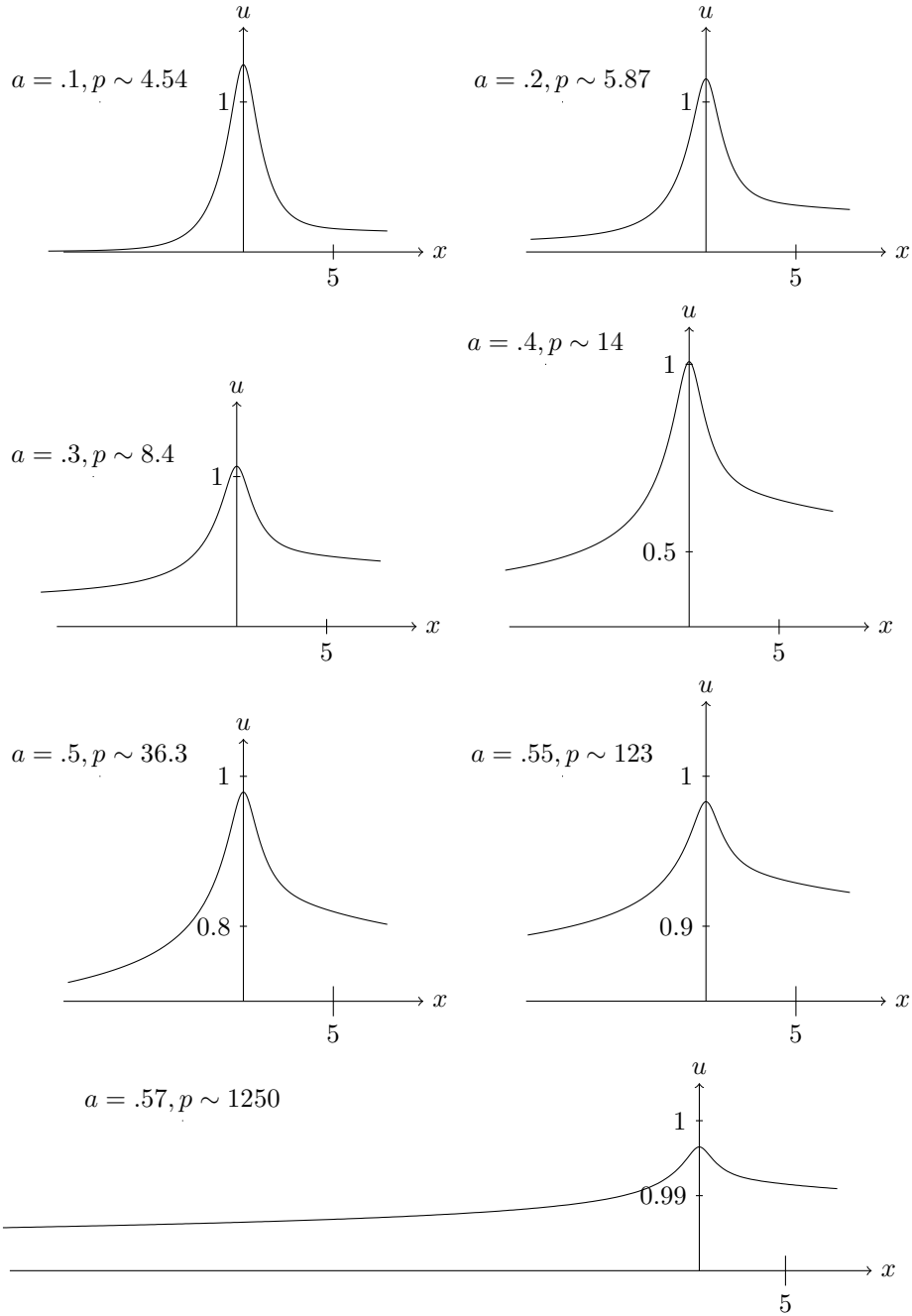
for  $x < -2a^{-1}$ .

## 8. A NUMERICAL SIMULATION

The selfsimilar solutions have been computed numerically by N. Strunk in his diploma thesis. The first curve shows  $1/p$  as a function of  $a$ . There is a small artefact near  $a = 0.1$ .



The next plots show the selfsimilar solution  $u$  as a function of  $x$  for various values of  $a$ .



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